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Probability Theory

On Cramér's theorem for capacities ☆

Sur théorème de Cramér pour capacités

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ABSTRACT

In this Note, our aim is to obtain Cramér's upper bound for capacities induced by sublinear expectations.

RÉSUMÉ

Dans cette Note, notre objet est d'obtenir la borne supérieure de Cramér pour les capacités induites par des espérances sous-linéaires.

Version française abrégée

Le théorème de Cramér est connu comme un résultat fondamental dans la théorie des grandes déviations. Il est très utile dans beaucoup de domaines.

Dans cette Note, nous nous intéressons à

$$\overline{E}[\cdot]=\sup_{Q\in\mathcal{P}}E_Q[\cdot],$$

où $\mathcal{P}$ est un ensemble de probabilités. En particulier, nous définissons $\overline{V}(A)=\overline{E}[I_A]=\sup_{Q\in\mathcal{P}}E_Q[I_A]$, $\forall A\in\mathcal{F}$. Bien évidemment, $\overline{V}$ est une capacité. L'objet principal de cette Note est d'obtenir la borne supérieure de Cramér pour cette capacité.

Voici notre résultat principal.

Théorème *(la borne supérieure de Cramér). Soient $\{X_n; n\geqslant 1\}$ une suite de variables aléatoires i.i.d. sous $\overline{E}[\cdot]$. Désignons $\overline{S}_n=\frac{1}{n}\sum_{i=1}^{n}X_i$, on a alors :*

pour tous les ensembles fermés $F\subset R$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\overline{V}(\overline{S}_n\in F)\leqslant-\inf_{x\in F}\overline{\wedge^*}(x), \qquad (*)$$

où $\overline{\wedge^*}(x):=\sup_{\lambda\in R}[\lambda x-\log\overline{E}[e^{\lambda X_1}]]$ est une fonction convexe de taux.

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1. Introduction

Cramér's theorem has been widely been known for a long time as a fundamental result in large deviations. It is very useful in many fields.

Since the paper (Artzner et al. [1]) on coherent risk measures, authors are more and more interested in sublinear expectations (and more generally, convex expectations, see Föllmer and Schied [3] and Frittelli and Rossaza Gianin [4]). By Peng [7], we know that a sublinear expectation $\hat{E}$ can be represented as the upper expectation of a set of linear expectations $\{E_\theta\colon \theta \in \Theta\}$, i.e., $\hat{E}[\cdot] = \sup_{\theta\in\Theta} E_\theta[\cdot]$. In most cases, this set is often treated as an uncertain model of probabilities $\{P_\theta\colon \theta \in \Theta\}$ and the notion of sublinear expectation provides a robust way to measure a risk loss X. In fact, nonlinear expectation theory provides many rich, flexible and elegant tools.

In this Note, we are interested in

$$\overline{E}[\cdot] = \sup_{Q\in\mathcal{P}} E_Q[\cdot],$$

where $\mathcal{P}$ is a set of probability measures. Specially, set $\overline{V}(A) = \overline{E}[I_A] = \sup_{Q\in\mathcal{P}} E_Q[I_A]$, $\forall A \in \mathcal{F}$. Obviously, $\overline{V}$ is a capacity. The main aim of this Note is to obtain Cramér's upper bound for the capacity $\overline{V}$.

This Note is organized as follows: in Section 2, we give some notions and lemmas that are useful in this Note. In Section 3, we give the main result including the proof.

2. Preliminaries

We present some preliminaries in the theory of sublinear expectations. More details of this section can be found in Peng [5–7].

Definition 2.1. Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We assume that all constants are in $\mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of our "random variables". A nonlinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \mapsto R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geqslant Y$ then $\hat{E}[X] \geqslant \hat{E}[Y]$.
(b) Constant preserving: $\hat{E}[c] = c$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a nonlinear expectation space (compare with a probability space $(\Omega, \mathcal{F}, P)$). We are mainly concerned with sublinear expectation where the expectation $\hat{E}$ satisfies also

(c) Sub-additivity: $\hat{E}[X] - \hat{E}[Y] \leqslant \hat{E}[X - Y]$.
(d) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$, $\forall \lambda \geqslant 0$.

If only (c) and (d) are satisfied, $\hat{E}$ is called a sublinear functional.

The following representation theorem for sublinear expectations is very useful (see Peng [6,7] for the proof):

Lemma 2.1. *Let $\hat{E}$ be a sublinear functional defined on $(\Omega, \mathcal{H})$, i.e.,* (c) *and* (d) *hold for $\hat{E}$. Then there exists a family $\{E_\theta\colon \theta \in \Theta\}$ of linear functionals on $(\Omega, \mathcal{H})$ such that*

$$\hat{E}[X] = \max_{\theta\in\Theta} E_\theta[X]. \tag{1}$$

If (a) *and* (b) *also hold, then E_θ are linear expectations for $\theta \in \Theta$. If we make furthermore the following assumption*: (H1) *For each sequence $\{X_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that $X_n(\omega) \downarrow 0$ for ω, we have $\hat{E}[X_n] \downarrow 0$. Then for each $\theta \in \Theta$, there exists a unique (σ-additive) probability measure P_θ defined on $(\Omega, \sigma(\mathcal{H}))$ such that*

$$E_\theta[X] = \int_\Omega X(\omega)\, \mathrm{d}P_\theta(\omega), \quad X \in \mathcal{H}. \tag{2}$$

In this Note, we are interested in the following sublinear expectation:

$$\overline{E}[\cdot] = \sup_{Q\in\mathcal{P}} E_Q[\cdot],$$

where $\mathcal{P}$ is a set of probability measures. Let Ω be a given set and let $\mathcal{F}$ be a σ-algebra. Define $\overline{V}(A) := \overline{E}[I_A] = \sup_{Q\in\mathcal{P}} E_Q[I_A]$, $\forall A \in \mathcal{F}$, then $\overline{V}$ is a capacity.

Let $C(R^n)$ denote the space of continuous functions defined on R^n.

Now we recall some important notions of sublinear expectations distributions (see Peng [5–7]).

Definition 2.2. Let X_1 and X_2 be two random variables in a sublinear expectation space $(\Omega, \mathcal{F}, \overline{E})$. They are called identically distributed, denoted by $X_1 \sim X_2$, if for $\varphi \in C(R)$, $\overline{E}[\varphi(X_1)]$ and $\overline{E}[\varphi(X_2)]$ exist, then we have

$$\overline{E}\big[\varphi(X_1)\big] = \overline{E}\big[\varphi(X_2)\big].$$

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{F}, \overline{E})$, a random vector $Y = (Y_1, \ldots, Y_n)$ is said to be independent of another random vector $X = (X_1, \ldots, X_m)$, if for $\varphi \in C(R^{m+n})$, $\overline{E}[\varphi(X, Y)]$ and $\overline{E}[\overline{E}[\varphi(x, Y)]_{x=X}]$ exist, then we have

$$\overline{E}\big[\varphi(X, Y)\big] = \overline{E}\big[\overline{E}\big[\varphi(x, Y)\big]_{x=X}\big].$$

3. Main result

In this section, firstly let us present some notations and assumptions that are used in the following:
Define

$$\underline{x} := -\overline{E}[-X];$$
$$\overline{\wedge}(\lambda) := \log \overline{E}\big[e^{\lambda X}\big], \quad \forall \lambda \in R;$$
$$\overline{\wedge^*}(x) := \sup_{\lambda \in R}\big[\lambda x - \overline{\wedge}(\lambda)\big], \quad \forall x \in R;$$
$$D_{\overline{\wedge}} := \big\{\lambda\colon \overline{\wedge}(\lambda) < \infty\big\}.$$

We always assume that

(H2) If $A_n \uparrow \Omega$, then $\overline{V}(A_n) \uparrow 1$.

Now we list our main result.

Theorem *(Cramér's upper bound). Let a random sequence $\{X_n;\ n \geqslant 1\}$ be identically distributed under $\overline{E}[\cdot]$. We also assume that each X_{n+1} is independent of $(X_1, \ldots, X_n)$ for $n = 1, 2, \ldots$ under $\overline{E}[\cdot]$. Denote $\overline{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then we have:*
For any closed set $F \subset R$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \overline{V}(\overline{S}_n \in F) \leqslant -\inf_{x\in F} \overline{\wedge^*}(x), \tag{$*$}$$

where $\overline{\wedge^*}(\cdot)$ is a convex rate function.

The proof of the Theorem is a straightforward adaptation of the classical arguments, e.g., in Lemma 2.2.5 and Theorem 2.2.3 of Ref. [2].
The following lemma states the properties of $\overline{\wedge^*}(\cdot)$ and $\overline{\wedge}(\cdot)$ that are needed to prove the Theorem:

Lemma 3.1.

(1) *$\overline{\wedge}$ is a convex function and $\overline{\wedge^*}$ is a convex rate function.*
(2) *If $D_{\overline{\wedge}} = \{0\}$, then $\overline{\wedge^*} \equiv 0$. If $\overline{\wedge}(\lambda) < \infty$ for some $\lambda > 0$, then $\underline{x} < \infty$ (possibly $\underline{x} = -\infty$), and for all $x \geqslant \underline{x}$,*

$$\overline{\wedge^*}(x) = \sup_{\lambda \geqslant 0}\big[\lambda x - \overline{\wedge}(\lambda)\big]. \tag{3}$$

Similarly, if $\overline{\wedge}(\lambda) < \infty$ for some $\lambda < 0$, then $\underline{x} > -\infty$ (possibly $\underline{x} = \infty$), and for all $x \leqslant \underline{x}$,

$$\overline{\wedge^*}(x) = \sup_{\lambda \leqslant 0}\big[\lambda x - \overline{\wedge}(\lambda)\big]. \tag{4}$$

(3) *When $\underline{x}$ is finite, $\overline{\wedge^*}(\underline{x}) = 0$, and always*

$$\inf_{x\in R} \overline{\wedge^*}(x) = 0. \tag{5}$$

Proof. (1) The convexity of $\overline{\wedge}$ follows by Hölder's inequality, since

$$\overline{\wedge}\big(\alpha\lambda_1 + (1-\alpha)\lambda_2\big) = \log \overline{E}\big[\big(e^{\lambda_1 X}\big)^{\alpha}\big(e^{\lambda_2 X}\big)^{1-\alpha}\big] \leqslant \log\big(\overline{E}\big[e^{\lambda_1 X}\big]^{\alpha}\overline{E}\big[e^{\lambda_2 X}\big]^{1-\alpha}\big) = \alpha\overline{\wedge}(\lambda_1) + (1-\alpha)\overline{\wedge}(\lambda_2)$$

for any $\alpha \in [0, 1]$. The convexity of $\overline{\wedge^*}$ follows from its notation, since

$$\alpha\overline{\wedge}^*(x_1) + (1-\alpha)\overline{\wedge}^*(x_2) = \sup_{\lambda\in R}\bigl[\alpha\lambda x_1 - \alpha\overline{\wedge}(\lambda)\bigr] + \sup_{\lambda\in R}\bigl[(1-\alpha)\lambda x_2 - (1-\alpha)\overline{\wedge}(\lambda)\bigr]$$
$$\geqslant \sup_{\lambda\in R}\bigl[\lambda\bigl(\alpha x_1 + (1-\alpha)x_2\bigr) - \overline{\wedge}(\lambda)\bigr] = \overline{\wedge}^*\bigl(\alpha x_1 + (1-\alpha)x_2\bigr).$$

Note that $\overline{\wedge}(0) = 0$, hence $\overline{\wedge}^*(x) \geqslant 0x - \overline{\wedge}(0) = 0$. In order to prove that $\overline{\wedge}^*$ is lower semicontinuous, fix a sequence $x_n \to x$. Then, for every $\lambda \in R$,

$$\liminf_{x_n\to x}\overline{\wedge}^*(x_n) \geqslant \liminf_{x_n\to x}\bigl[\lambda x_n - \overline{\wedge}(\lambda)\bigr] = \lambda x - \overline{\wedge}(\lambda).$$

So $\liminf_{x_n\to x}\overline{\wedge}^*(x_n) \geqslant \overline{\wedge}^*(x)$. By the definition of rate function (see Dembo and Zeitouni [2, p. 4]), we know that $\overline{\wedge}^*$ is a rate function.

(2) If $D_{\overline{\wedge}} = \{0\}$, then $\overline{\wedge}^*(x) = \overline{\wedge}(0) = 0$ for all $x \in R$. If $\overline{\wedge}(\lambda) = \log\overline{E}[e^{\lambda X}] < \infty$ for some $\lambda > 0$, then $\overline{E}[X^+] = \sup_{Q\in\mathcal{P}}\int_\Omega X^+\,dQ \leqslant \frac{\overline{E}[e^{\lambda X}]}{\lambda} < \infty$, implying that $\underline{x} < \infty$ (possibly $\underline{x} = -\infty$). Now, for all $\lambda \in R$, by Jensen's Inequality,

$$\overline{\wedge}(\lambda) = \log\overline{E}\bigl[e^{\lambda X}\bigr] \geqslant \overline{E}\bigl[\log e^{\lambda X}\bigr] = \overline{E}[\lambda X] \geqslant \lambda\underline{x}.$$

If $\underline{x} = -\infty$, then $\overline{\wedge}(\lambda) = \infty$ for $\lambda < 0$, and (3) trivially holds. When $\underline{x}$ is finite, it follows from the preceding inequality that $\overline{\wedge}^*(\underline{x}) = 0$. In this case, for every $x \geqslant \underline{x}$ and every $\lambda < 0$, $\lambda x - \overline{\wedge}(\lambda) \leqslant \lambda\underline{x} - \overline{\wedge}(\lambda) \leqslant \overline{\wedge}^*(\underline{x}) = 0$, so (3) holds. In a similar manner, we can prove the rest of (2).

(3) It remains to prove that $\inf_{x\in R}\overline{\wedge}^*(x) = 0$. This is already established for $D_{\overline{\wedge}} = \{0\}$, in which case $\overline{\wedge}^* \equiv 0$, and when $\underline{x}$ is finite, in which case, as shown before, $\overline{\wedge}^*(\underline{x}) = 0$. Now, consider the case when $\underline{x} = -\infty$ while $\overline{\wedge}(\lambda) < \infty$ for some $\lambda > 0$. Then, by Chebycheff's inequality and (3), we have

$$\log\overline{V}\bigl(X \in [x,\infty)\bigr) \leqslant \inf_{\lambda\geqslant 0}\log\overline{E}\bigl[e^{\lambda(X-x)}\bigr] = -\sup_{\lambda\geqslant 0}\bigl\{\lambda x - \overline{\wedge}(\lambda)\bigr\} = -\overline{\wedge}^*(x).$$

Thus, applying (H2),

$$\lim_{x\to-\infty}\overline{\wedge}^*(x) \leqslant \lim_{x\to-\infty}\bigl[-\log\overline{V}\bigl(X\in[x,\infty)\bigr)\bigr] = 0,$$

and (5) follows. The last remaining case, that of $\underline{x} = \infty$ while $\overline{\wedge}(\lambda) < \infty$ for some $\lambda < 0$ can be proved in a similar manner of the above. □

Remark 3.1. By the proof of Lemma 3.1(2), we know that if $\underline{x} < \infty$, then for all $x \geqslant \underline{x}$, (3) also holds and if $\underline{x} > -\infty$, then for all $x \leqslant \underline{x}$, (4) also holds.

Proof of Theorem. Let F be a non-empty closed set. When $M_F := \inf_{x\in F}\overline{\wedge}^*(x) = 0$, $(*)$ trivially holds. Assume that $M_F > 0$. For all x and every $\lambda \geqslant 0$, by Chebycheff's inequality and Definitions 2.2 and 2.3, we have

$$\overline{V}\bigl(\overline{S}_n \in [x,\infty)\bigr) \leqslant \overline{E}\bigl[e^{n\lambda(\overline{S}_n - x)}\bigr] = e^{-n\lambda x}\prod_{i=1}^n\overline{E}\bigl[e^{\lambda X_i}\bigr] = e^{-n[\lambda x - \overline{\wedge}(\lambda)]}. \tag{6}$$

Therefore, if $\underline{x} < \infty$, then by (3), for every $x > \underline{x}$,

$$\overline{V}\bigl(\overline{S}_n \in [x,\infty)\bigr) \leqslant e^{-n\overline{\wedge}^*(x)}. \tag{7}$$

By a similar argument, if $\underline{x} > -\infty$ and $x < \underline{x}$, then

$$\overline{V}\bigl(\overline{S}_n \in (-\infty,x]\bigr) \leqslant e^{-n\overline{\wedge}^*(x)}. \tag{8}$$

First, consider the case of $\underline{x}$ finite. Then $\overline{\wedge}^*(\underline{x}) = 0$, and because by assumption $M_F > 0$, $\underline{x}$ must be contained in the open set F^c. Let (x_-, x_+) be the union of all open intervals $(a,b) \in F^c$ that contain $\underline{x}$. Note that $x_- < x_+$ and that either x_- or x_+ must be finite since F is non-empty. If x_- is finite, then $x_- \in F$ and consequently $\overline{\wedge}^*(x_-) \geqslant M_F$. Likewise, $\overline{\wedge}^*(x_+) \geqslant M_F$ whenever x_+ is finite. Applying (7) and (8), we have

$$\overline{V}(\overline{S}_n \in F) \leqslant \overline{V}\bigl(\overline{S}_n \in (-\infty,x_-]\bigr) + \overline{V}\bigl(\overline{S}_n \in [x_+,\infty)\bigr) \leqslant 2e^{-nM_F}.$$

So as $n \to \infty$, $(*)$ holds.

Suppose now that $\underline{x} = -\infty$. Then since $\overline{\wedge}^*$ is nondecreasing, it follows from (5) that $\lim_{x\to-\infty}\overline{\wedge}^*(x) = 0$, and hence $x_+ = \inf\{x\colon\ x \in F\}$ is finite for otherwise $M_F = 0$. Since F is a closed set, $x_+ \in F$ and consequently $\overline{\wedge}^*(x_+) \geqslant M_F$. Note that $F \subset [x_+,\infty)$, hence $(*)$ holds by applying (7) for $x = x_+$.

The case of $\underline{x} = \infty$ is handled analogously. □

Remark 3.2. (1) A close inspection of the proof reveals that the assumptions (A) (i.e., the random sequence $\{X_n;\ n \geqslant 1\}$ is identically distributed) and (B) (i.e., each X_{n+1} is independent of $(X_1, \ldots, X_n)$ for $n = 1, 2, \ldots$) can be replaced by the weaker assumptions (A′) that the random sequence $\{X_n;\ n \geqslant 1\}$ is identically distributed with respect to the continuous functions $\varphi_\lambda(x) = \mathrm{e}^{\lambda x}$, $\forall\lambda \in R$ and $\varphi(x) = -x$ and (B′) that each X_{n+1} is independent of $(X_1, \ldots, X_n)$ with respect to the continuous functions $\varphi_\lambda(x_1, \ldots, x_n, x_{n+1}) = \mathrm{e}^{\lambda \sum_{i=1}^{n+1} x_i}$, $\forall\lambda \in R$, $n = 1, 2, \ldots.$

(2) Under the assumptions of Theorem, for any open set $G \subset R$, does $\liminf_{n\to\infty} \frac{1}{n} \log \overline{V}(\bar{S}_n \in G) \geqslant -\inf_{x\in G} \overline{\wedge^*}(x)$ hold? The answer is negative. Now we give a counterexample. Let $\mathcal{P} = \{P_1, P_2\}$. Assume that $X_i \equiv M$, $M > 0$, $i = 1, 2, \ldots,$ in probability measure P_1 and $X_i \equiv 2M$, $i = 1, 2, \ldots,$ in probability measure P_2. Obviously, (A′) and (B′) are satisfied. We choose $G = (M, 2M)$. By a simple computation, $\frac{1}{n} \log \overline{V}(\bar{S}_n \in G) = -\infty$, for each n, but $\inf_{x\in G} \overline{\wedge^*}(x) = 0$. Hence for $G = (M, 2M)$, $\liminf_{n\to\infty} \frac{1}{n} \log \overline{V}(\bar{S}_n \in G) < -\inf_{x\in G} \overline{\wedge^*}(x)$.

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