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Algebraic Geometry

Motivic decomposability of generalized Severi–Brauer varieties

Décomposabilité motivique des variétés de Severi–Brauer généralisées

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ABSTRACT

Let F be an arbitrary field. Let p be a positive prime number and D a central division F-algebra of degree p^n, with $n \geqslant 1$. We write $SB(p^m, D)$ for the generalized Severi–Brauer variety of right ideals in D of reduced dimension p^m for $m = 0, 1, \dots, n-1$. We note by $M(SB(p^m, D))$ the Chow motive with coefficients in $\mathbb{F}_p$ of the variety $SB(p^m, D)$. It was proven by Nikita Karpenko that this motive is indecomposable for any prime p and $m = 0$ and for $p = 2$, $m = 1$. We prove decomposability of $M(SB(p^m, D))$ in all the other cases.

RÉSUMÉ

Soient F un corps arbitraire, p un nombre premier positif et D une F-algèbre de division de degré p^n. On écrit $SB(p^m, D)$ pour la variété de Severi–Brauer généralisée des idéaux à droite de dimension réduite p^m, $m = 0, 1, \dots, n-1$. On note par $M(SB(p^m, D))$ le motif de Chow à coefficients dans $\mathbb{F}_p$ de la variété $SB(p^m, D)$. Il a été demontré par Nikita Karpenko que ce motif est indecomposable pour tout nombre premier p arbitraire et $m = 0$ et pour $p = 2$, $m = 1$. Nous montrons la décomposabilité de $M(SB(p^m, D))$ dans tous les autres cas.

1. Chow motives with finite coefficients

Our basic reference for Chow groups and Chow motives (including notations) is [3]. We fix an associative unital commutative ring Λ and for a variety (i.e., a separated scheme of finite type over a field) X we write $\mathrm{Ch}(X)$ for its Chow group with coefficients in Λ (while we write $\mathrm{CH}(X)$ for its integral Chow group). Our category of motives is the category $\mathrm{CM}(F, \Lambda)$ of graded Chow motives with coefficients in Λ, [3, definition of §64]. By a sum of motives we always mean the direct sum. We also write Λ for the motive $M(\operatorname{Spec} F) \in \mathrm{CM}(F, \Lambda)$. A Tate motive is the motive of the form $\Lambda(i)$ with i an integer.

Let X be a smooth complete variety over F and let M be a motive. We call M split if it is a finite sum of Tate motives. We call X split, if its integral motive $M(X) \in \mathrm{CM}(F, \mathbb{Z})$ (and therefore the motive of X with an arbitrary coefficient ring Λ) is split. We call M or X geometrically split, if it splits over a field extension of F. Over an extension of F the geometrically split motive M becomes isomorphic to a finite sum of Tate motives. We write $\operatorname{rk} M$ for the number of the summands in this decomposition (this number do not depend on the choice of the splitting field extension).

We say that X satisfies the nilpotence principle, if for any field extension E/F and any coefficient ring Λ, the kernel of the change of field homomorphism $\operatorname{End}(M(X)) \to \operatorname{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous (under

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an action of a semisimple affine algebraic group) variety is geometrically split and satisfies the nilpotence principle [3, Theorem 92.4 with Remark 92.3].

A complete decomposition of an object in an additive category is a finite direct sum decomposition with indecomposable summands. We say that the Krull–Schmidt principle holds for a given object of a given additive category, if every direct sum decomposition of the object can be refined to a complete one (in particular, a complete decomposition exists) and there is only one (up to a permutation of the summands) complete decomposition of the object. We have the following theorem:

Theorem 1.1. *(See [2, Theorem 3.6 of Chapter I].) Assume that the coefficient ring Λ is finite. The Krull–Schmidt principle holds for any shift of any summand of the motive of any geometrically split F-variety satisfying the nilpotence principle.*

Lemma 1.2. *Assume that the coefficient ring Λ is finite. Let X be a variety satisfying the nilpotence principle. Let $f \in \mathrm{End}(M(X))$ and $1_E = f_E \in \mathrm{End}(M(X)_E)$ for some field extension E/F. Then $f^n = 1$ for some positive integer n.*

Proof. Since X satisfies the nilpotence principle, we have $f = 1 + \varepsilon$, where ε is nilpotent. Let n be a positive integer such that $\varepsilon^n = 0 = n\varepsilon$. Then $f^{n^n} = (1+\varepsilon)^{n^n} = 1$ because the binomial coefficients $\binom{n^n}{i}$ for $i < n$ are divisible by n. □

2. Motivic decomposability of generalized Severi–Brauer varieties

Let p be a positive prime integer. The coefficient ring Λ is $\mathbb{F}_p$ in this section. Let F be a field. Let D be a central division F-algebra of degree p^n. We write $SB(p^m, D)$ for the generalized Severi–Brauer variety of right ideals in D of reduced dimension p^m for $m = 0, 1, \ldots, n-1$.

Lemma 2.1. *Let E/F be a splitting field extension for $X = SB(1, D)$. Then the subgroup of F-rational cycles in $\mathrm{Ch}_{\dim X}(X_E \times X_E)$ is generated by a diagonal class.*

Proof. We write $\overline{\mathrm{Ch}}(X)$ for the image of the homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(X_E)$. By [6, Proposition 2.1.1], we have $\overline{\mathrm{Ch}}^i(X) = 0$ for $i > 0$. Since the (say, first) projection $X^2 \to X$ is a projective bundle, we have a (natural with respect to the base field change) isomorphism $\mathrm{Ch}_{\dim X}(X^2) \simeq \mathrm{Ch}(X)$. Passing to $\overline{\mathrm{Ch}}$, we get an isomorphism $\overline{\mathrm{Ch}}_{\dim X}(X^2) \simeq \overline{\mathrm{Ch}}(X) = \overline{\mathrm{Ch}}^0(X)$ showing that $\dim_{\mathbb{F}_p} \overline{\mathrm{Ch}}_{\dim X}(X^2) = 1$. Since the diagonal class in $\overline{\mathrm{Ch}}_{\dim X}(X^2)$ is non-zero, it generates all the group. □

Corollary 2.2. *(Cf. [6, Theorem 2.2.1].) The motive with coefficients in $\mathbb{F}_p$ of the Severi–Brauer variety $X = SB(1, D)$ is indecomposable.*

Proof. To prove that our motive is indecomposable it is enough to show that $\mathrm{End}(M(X)) = \mathrm{Ch}_{\dim X}(X \times X)$ does not contain non-trivial projectors. Let $\pi \in \mathrm{Ch}_{\dim X}(X \times X)$ be a projector. By Lemma 2.1, π_E is zero or equal to 1_E. Since X satisfies the nilpotence principle, π is nilpotent in the first case, but also idempotent, therefore π is zero. Lemma 1.2 gives us $\pi = 1$ in the second case. □

Nikita Karpenko has recently proved the motivic indecomposability of generalized Severi–Brauer varieties also in the case $p = 2$, $m = 1$.

Theorem 2.3. *(Cf. [8, Theorem 4.2].) Let D be a central division F-algebra of degree 2^n with $n \geqslant 1$. Then the motive with coefficients in $\mathbb{F}_2$ of the variety $SB(2, D)$ is indecomposable.*

Taking into account Corollary 2.2, Theorem 2.3 and the fact that any generalized Severi–Brauer variety $SB(p^m, D)$ is p-incompressible [8, Theorem 4.3] (this condition is weaker than motivic indecomposability), one can expect that the Chow motive with coefficients in $\mathbb{F}_p$ of any generalized Severi–Brauer variety $SB(p^m, D)$ is indecomposable. But the following theorem gives us the opposite answer:

Theorem 2.4. *Let D be a central division F-algebra of degree p^n with $n \geqslant 1$. Then the motive with coefficients in $\mathbb{F}_p$ of the variety $SB(p^m, D)$ is decomposable for $p = 2$, $1 < m < n$ and for $p > 2$, $0 < m < n$. In these cases $M(SB(1, D))(k)$ is a summand of $M(SB(p^m, D))$ for $2 \leqslant k \leqslant p^n - p^m$.*

Proof. We use the notations: $X = SB(1, D)$, $Y = SB(p^m, D)$, $d = \dim(SB(1, D)) = p^n - 1$, $r = p^n - p^m$. Let $E = F(X)$, then E/F is a splitting field extension for the variety X (and also for Y). We will show that $M(X)(k)$ is a summand of $M(Y)$. By Lemma 1.2 it suffices to construct two F-rational morphisms

$$\alpha : M(X_E)(k) \to M(Y_E) \quad \text{and} \quad \beta : M(Y_E) \to M(X_E)(k)$$

satisfying $\beta \circ \alpha = 1 \in \mathrm{End}(M(X_E)(k)) = \mathrm{Ch}_d(X_E \times X_E)$. By Lemma 2.1 we can replace condition $\beta \circ \alpha = 1$ by $\beta \circ \alpha \neq 0$.

Let *Tav* be the tautological vector bundle on X. The product $X \times Y$ considered over X (via the first projection) is isomorphic (as a scheme over X) to the Grassmann bundle $G_r(Tav)$ of r-dimensional subspaces in *Tav* (cf. [5, Proposition 4.3]). Let T be the tautological r-dimensional vector bundle on $G_r(Tav)$. Over the field E the algebra D becomes isomorphic to $\mathrm{End}_E(V)$ for some E-vector space V of dimension $d+1=p^n$. We have $X_E \simeq \mathbb{P}^d(V)$ and $Y_E \simeq G_{p^m}(V)$. Let T_1 and T_{p^m} be the tautological bundles of rank 1 and p^m on X_E and Y_E respectively. Then we have an isomorphism (cf. [5, Proposition 4.3]): $T_E \simeq T_1 \boxtimes (-T_{p^m})^\vee$ (here we lift the bundles T_1 and T_{p^m} on $X_E \times Y_E$). Let $h = c_1(T_1) \in \mathrm{Ch}^1(X_E)$ (then $-h$ is a hyperplane class in $\mathrm{Ch}^1(X_E)$) and $c_i = c_i((-T_{p^m})^\vee) \in \mathrm{Ch}^i(Y_E)$. Then by [4, Remark 3.2.3(b)]

$$c_t(T_E) = c_t\big(T_1 \boxtimes (-T_{p^m})^\vee\big) = \sum_{i=0}^{r} \big(1 + (h \times 1)t\big)^{r-i} (1 \times c_i) t^i.$$

It follows from the conditions of the theorem that the binomial coefficients $\binom{p^n-p^m}{2}, \binom{p^n-p^m}{p^m-1}$ are divisible by p and $\binom{p^n-p^m-1}{p^m-2} \equiv (-1)^{p^m-2} \bmod p$. Therefore

$$c_1(T_E) = 1 \times c_1, \qquad c_2(T_E) = -h \times c_1 + 1 \times c_2, \qquad c_{p^m-1}(T_E) = (-1)^{p^m-2} h^{p^m-2} \times c_1 + \cdots,$$

where "$\cdots$" stands for a linear combination of only those terms whose second factor has codimension > 1. For the top Chern class we have: $c_r(T_E) = \sum_{i=0}^{r} h^{r-i} \times c_i$. Let $\beta_1 = c_r(T_E) c_{p^m-1}(T_E) c_2(T_E) c_1(T_E)^{k-2} = (-h)^d \times c_1^k + \cdots = x \times c_1^k + \cdots$, where "$\cdots$" stands for a linear combination of only those terms whose second factor has codimension $> k$ and where x is the class of a rational point in $\mathrm{Ch}(X_E)$. We take $\beta = \beta_1^t$, where β_1^t is the transpose of β_1. Since the bundle T is defined over F, the morphism $\beta \in \mathrm{Ch}_{\dim Y - k}(Y_E \times X_E) = \mathrm{Hom}(M(Y_E), M(X_E)(k))$ is F-rational.

By [4, Example 14.6.6] the cycle c_1^k is non-zero. Let $a \in \mathrm{Ch}(Y_E)$ be the element dual to c_1^k with respect to the bilinear form $\mathrm{Ch}(Y_E) \times \mathrm{Ch}(Y_E) \to \mathbb{F}_p$, $(x_1, x_2) \mapsto \deg(x_1 \cdot x_2)$. The pull-back homomorphism $f: \mathrm{Ch}(X \times Y) \to \mathrm{Ch}(Y_{F(X)}) = \mathrm{Ch}(Y_E)$ with respect to the morphism $Y_{F(X)} = (\mathrm{Spec}\, F(X)) \times Y \to X \times Y$ given by the generic point of X is surjective by [3, Corollary 57.11]. Let $\alpha' \in \mathrm{Ch}(X \times Y)$ be a cycle whose image in $\mathrm{Ch}(Y_E)$ under the surjection f is a. We define α as α'_E and we have $\alpha = 1 \times a + \cdots$, where "$\cdots$" stands for a linear combination of only those elements whose first factor is of positive codimension and where $1 = [X_E]$. Then $\beta \circ \alpha = 1 \times x + \cdots$, where "$\cdots$" stands for a linear combination of only those terms whose first factor is of positive codimension. It follows that $\beta \circ \alpha \neq 0$. □

Remark 2.5. Theorem 2.4 also gives us some information about the integral motive of the variety $SB(p^m, D)$. Indeed, according to [9, Corollary 2.7] the decomposition of $M(SB(p^m, D))$ with coefficients in $\mathbb{F}_p$ lifts (and in a unique way) to the coefficients $\mathbb{Z}/p^N\mathbb{Z}$ for any $N \geqslant 2$. Then by [9, Theorem 2.16] it lifts to $\mathbb{Z}$ (uniquely for $p=2$ and $p=3$ and non-uniquely for $p>3$). See also Remark 2.8.

Remark 2.6. Let l be an integer such that $0 < l < p^n$ and $\gcd(l, p) = 1$. The complete decomposition of the motive $M(SB(l, D))$ with coefficients in $\mathbb{F}_p$ is described in [1, Proposition 2.4].

Example 2.7. As an application of Theorem 2.4 we describe the complete motivic decomposition of $SB(4, D)$ with coefficients in $\mathbb{F}_2$ for a division algebra D of degree 8. We denote by M the motive $M(SB(1, D))$. By Theorem 2.4, the motives $M(2)$, $M(3)$, $M(4)$ and by duality $M(7)$, $M(6)$, $M(5)$ are direct summands of $M(SB(4, D))$. We have $M(SB(4, D)) = M(2) \oplus \cdots \oplus M(7) \oplus N$ for some motive N. Assume that N is decomposable. Then by [8, Theorem 3.8], and Theorems 2.2, 2.3, the motive N has an indecomposable summand which is some shift of either M or $M(SB(2, D))$. But the second case is impossible because $70 = \binom{8}{4} = \mathrm{rk}\, M(SB(4, D)) < 6\, \mathrm{rk}\, M + \mathrm{rk}\, M(SB(2, D)) = 6 \cdot 8 + \binom{8}{2} = 76$ (see [8, Example 2.18] for the computations of ranks). Therefore $M(i)$ is a summand of N for some integer i. According to [7, Corollary 10.19], we can write the complete decomposition of N over the function field $L = F(SB(4, D))$:

$$N_L = \mathbb{F}_2 \oplus \widetilde{M}(1) \oplus M\big(SB(2, C)\big)(4) \oplus M\big(SB(2, C)\big)(8) \oplus \widetilde{M}(12) \oplus \mathbb{F}_2(16),$$

where C is a central division L-algebra (of degree 4) Brauer-equivalent to D_L and where $\widetilde{M} = M(SB(1, C))$. It follows from this decomposition that the motive $M(i)_L = \widetilde{M}(i) \oplus \widetilde{M}(i+4)$ cannot be a summand of N_L. We have a contradiction. Therefore the motive N is indecomposable and we have a complete motivic decomposition of $SB(4, D)$ with coefficients in $\mathbb{F}_2$:

$$M\big(SB(4, D)\big) = N \oplus M(2) \oplus M(3) \oplus M(4) \oplus M(5) \oplus M(6) \oplus M(7). \tag{1}$$

Remark 2.8. We have the same decomposition as (1) for the integral motive of the variety $SB(4, D)$. To show this one can apply [9, Corollary 2.7] and then [9, Theorem 2.16].

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References

[1] B. Calmès, V. Petrov, N. Semenov, K. Zainoulline, Chow motives of twisted flag varieties, Compos. Math. 142 (4) (2006) 1063–1080.
[2] V. Chernousov, A. Merkurjev, Motivic decomposition of projective homogeneous varieties and the Krull–Schmidt theorem, Transform. Groups 11 (3) (2006) 371–386.
[3] R. Elman, N. Karpenko, A. Merkurjev, The Algebraic and Geometric Theory of Quadratic Forms, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008.
[4] W. Fulton, Intersection Theory, second edition, Springer, Berlin, 1998.
[5] O. Izhboldin, N. Karpenko, Some new examples in the theory of quadratic forms, Math. Z. 234 (2000) 647–695.
[6] N.A. Karpenko, Grothendieck chow motives of Severi–Brauer varieties, Algebra i Analiz 7 (4) (1995) 196–213.
[7] N.A. Karpenko, Cohomology of relative cellular spaces and of isotropic flag varieties, Algebra i Analiz 12 (1) (2000) 3–69.
[8] N. Karpenko, Upper motives of algebraic groups and incompressibility of Severi–Brauer varieties, Linear Algebraic Groups and Related Structures (preprint server) 333 (2009, Apr. 2).
[9] V. Petrov, N. Semenov, K. Zainoulline, J-invariant of linear algebraic groups, Ann. Sci. École Norm. Sup. (4) 41 (2008) 1023–1053.