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Asymptotics of the KPP minimal speed within large drift

Asymptotiques de la vitesse minimale KPP dans le cas d'une grande advection

Mohammad El Smaily, Stéphane Kirsch

Department of Mathematics, University of British Columbia & Pacific Institute for the Mathematical Sciences, 1984 Mathematics Road, V6T 1Z2, Vancouver, BC, Canada

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ABSTRACT

This Note is concerned with the asymptotic behavior of the minimal KPP speed of propagation for reaction–advection–diffusion equations with a large drift Mq (where q is the advection). We first give the limit of the speed as $M \to +\infty$ in any space dimension N. Then, we give the necessary and sufficient condition that the advection field should satisfy so that the speed acts as $O(M)$ as $M \to +\infty$.

RÉSUMÉ

Dans cette Note on étudie le comportement asymptotique de la vitesse minimale de propagation des fronts progressifs pulsatoires satisfaisant une équation de réaction–advection–diffusion dans le cas d'une grande advection Mq (où q est l'advection). On donne la valeur limite de la vitesse lorsque $M \to +\infty$ dans un espace de dimension N quelconque. Pour le cas $N = 2$ on donne une condition nécessaire et suffisante pour que la vitesse se comporte comme $O(M)$ pour $M \to +\infty$.

Version française abrégée

Pour chaque $M > 0$, nous considérons l'équation de réaction–advection–diffusion suivante :

$$\begin{cases} u_t = \nabla \cdot \big(A(z)\nabla u\big) + Mq(z) \cdot \nabla u + f(z,u), & t \in \mathbb{R},\ z \in \Omega, \\ \nu \cdot A\nabla u = 0 \quad \text{sur } \mathbb{R} \times \partial\Omega, \end{cases} \tag{1}$$

où ν est la normale unitaire extérieure sur $\partial\Omega$ si $\partial\Omega \neq \emptyset$. Le domaine Ω est un sous-ensemble connexe de $\mathbb{R}^N$ de classe C^3 pour lequel il existe $1 \leqslant d \leqslant N$, $L_1, \dots, L_d$ positifs et $R > 0$ tels que

$$\forall (x,y) \in \Omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-d}, \quad |y| \leqslant R, \text{ et } \forall k = (k_1, \dots, k_d, 0, \dots, 0) \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z} \times \{0\}^{N-d},\ \Omega = \Omega + k.$$

On note C la cellule de périodicité de Ω définie par :

$$C = \big\{(x,y) \in \Omega;\ x_1 \in (0, L_1), \dots, x_d \in (0, L_d)\big\}.$$

E-mail addresses: elsmaily@math.ubc.ca (M. El Smaily), kirsch@math.ubc.ca (S. Kirsch).

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Dans ce cadre périodique, un champ $v : \Omega \to \mathbb{R}^N$ est dit L-periodique en x si $w(x+k, y) = w(x, y)$ p.p. dans Ω quel que soit $k = (k_1, \dots, k_d) \in \prod_{i=1}^d L_i\mathbb{Z}$. La diffusion $A(x, y) = (A_{ij}(x, y))_{1\leqslant i,j\leqslant N}$ dans l'équation (7) est un champ matriciel de classe $C^{2,\delta}(\overline{\Omega})$ (avec $\delta > 0$) L-périodique en x et vérifie :

$$\exists 0 < \alpha_1 \leqslant \alpha_2,\ \forall (x, y) \in \Omega,\ \forall \xi \in \mathbb{R}^N, \quad \alpha_1|\xi|^2 \leqslant \sum_{1\leqslant i,j\leqslant N} A_{ij}(x, y)\xi_i\xi_j \leqslant \alpha_2|\xi|^2.$$

L'advection $q(x, y) = (q_1(x, y), \dots, q_N(x, y))$ est un champ vectoriel L-périodique en x et de classe $C^{1,\delta}(\overline{\Omega})$ $(\delta > 0)$ satisfaisant :

$$\nabla \cdot q = 0 \quad \text{dans } \overline{\Omega},\ q \cdot \nu = 0 \text{ sur } \partial\Omega \text{ (quand } \partial\Omega \neq \emptyset),\ \text{ et } \forall 1 \leqslant i \leqslant d,\ \int_C q_i \,dx\,dy = 0. \tag{2}$$

La partie non linéaire $f = f(x, y, u)$ est une fonction positive de classe $C^{1,\delta}(\overline{\Omega} \times [0, 1])$, L-périodique en x telle que $f(x, y, 0) = f(x, y, 1) = 0$ pour tout $(x, y) \in \overline{\Omega}$, et

$$\begin{cases} \exists \rho \in (0, 1),\ \forall (x, y) \in \overline{\Omega},\ \forall 1-\rho \leqslant s \leqslant s' \leqslant 1,\ f(x, y, s) \geqslant f(x, y, s'), \\ \forall (x, y) \in \overline{\Omega}, \quad \zeta(x, y) := f'_u(x, y, 0) = \lim_{u\to 0^+} \frac{f(x,y,u)}{u} > 0. \end{cases} \tag{3}$$

On suppose aussi que la réaction f satisfait la condition « KPP » (d'après Kolmogorov, Petrovsky et Piskunov [8]),

$$\forall (x, y, s) \in \overline{\Omega} \times (0, 1), \quad 0 < f(x, y, s) \leqslant f'_u(x, y, 0)s = \zeta(x, y)s. \tag{4}$$

Un exemple de cette non linéarité est la fonction « homogène » $f(u) = u(1-u)$ sur $(0, 1)$.

On s'intéresse au phénomène de propagation des fronts progressifs pulsatoires pour l'équation (7). On fixe une direction unitaire $e \in \mathbb{R}^d$ et nous notons $\tilde{e} = (e, 0, \dots, 0) \in \mathbb{R}^N$. Un front progressif pulsatoire qui se propage *dans la direction de* $-e$ *avec une vitesse* c est une solution $u(t, x, y)$ de (7) de la forme $u(t, x, y) = \phi(x\cdot e + ct, x, y)$ où la fonction ϕ est L-périodique en x et vérifie les conditions limites $\lim_{s\to-\infty} \phi(s, x, y) = 0$ et $\lim_{s\to+\infty} \phi(s, x, y) = 1$ uniformément en $(x, y) \in \overline{\Omega}$. *D'après les résultats de* [1] *et* [2], *il existe une valeur critique notée* $c^*_{\Omega,A,Mq,f}(e) > 0$, *appellée vitesse minimale KPP, telle qu'il existe un front progressif pulsatoire pour l'équation* (7) *avec une vitesse* c *si et seulement si* $c \geqslant c^*_{\Omega,A,Mq,f}(e) > 0$. De plus, selon [3], le terme paramétrique $c^*_{\Omega,A,Mq,f}(e)/M$ reste borné indépendamment de $M \geqslant 1$. On détermine, dans [7], la limite de $c^*_{\Omega,A,Mq,f}(e)/M$ lorsque $M \to +\infty$. Dans le théorème suivant, on donne des détails sur la limite $c^*_{\Omega,A,Mq,f}(e)/M$ pour la dimension $N = 2$:

Lemme 0.1 *(Trajectoires périodique non bornées de q). On suppose ici que la dimension est* $N = 2$ *et donc* $d \in \{1, 2\}$. *Soit* $T(x)$ *une trajectoire périodique non bornée de q dans* Ω *passant par x. C'est-à-dire, il existe* $\mathbf{a} \in L_1\mathbb{Z} \times L_2\mathbb{Z} \setminus \{0\}$ *(resp.* $L_1\mathbb{Z} \times \{0\} \setminus \{0\}$*) pour* $d = 2$ *(resp.* $d = 1$*) tel que* $T(x) = T(x) + \mathbf{a}$. *Dans ce cas, on dit que* $T(x)$ *est* $\mathbf{a}$*-périodique. Alors, toutes les autres trajectoires non bornées périodiques* $T(y)$ *de q sont aussi* $\mathbf{a}$*-périodique. De plus,* $\mathbf{a} = L_1e_1$ *si* $d = 1$ *et donc, dans le cas* $d = 1$, *toutes les trajectoires périodiques sont* L_1e_1*-périodiques.*

Théorème 0.2. *On suppose que* $N = 2$ *et q,* Ω*, A et f satisfont les hypothèses mentionnés ci-dessus (avec* $N = 2$*). Alors,*

(i) *S'il n'existe pas de trajectoire périodique non bornée de q, alors* $\lim_{M\to+\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} = 0$, *quelle que soit la direction unitaire e.*
(ii) *S'il existe une trajectoire périodique non bornée* $T(x)$ *de q dans* Ω *(à laquelle on peut donc associer une période* $\mathbf{a} \in \mathbb{R}^2$*), alors*

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} > 0 \quad \Longleftrightarrow \quad \tilde{e} \cdot \mathbf{a} \neq 0. \tag{5}$$

De plus, dans le cas où $d = 1$, *le Lemme* 1.3 *implique que* $\tilde{e} \cdot \mathbf{a} = \pm L_1 \neq 0$. *Utilisant* (16), *on peut alors conclure que, pour* $d = 1$,

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} > 0 \quad \Longleftrightarrow \quad \big(\textit{Il existe une trajectoire périodique non bornée } T(x) \textit{ de } q \textit{ dans } \Omega\big). \tag{6}$$

1. Asymptotics of the minimal speed within large drift in any dimension *N* with more details in the case *N* = 2

For each $M > 0$, we consider the reaction-advection-diffusion equation:

$$\begin{cases} u_t = \nabla \cdot \big(A(z)\nabla u\big) + Mq(z)\cdot\nabla u + f(z, u), \quad t \in \mathbb{R},\ z \in \Omega, \\ \nu \cdot A\nabla u = 0 \quad \text{on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{7}$$

where ν stands for the unit outward normal on $\partial\Omega$ whenever it is nonempty.

The domain Ω is a C^3 nonempty connected open subset of $\mathbb{R}^N$ such that for some integer $1 \leqslant d \leqslant N$, there exist $L_1, \dots, L_d$ positive real numbers such that

$$\begin{cases} \exists R \geqslant 0,\ \forall (x,y) \in \Omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-d}, |y| \leqslant R, \\ \forall (k_1,\dots,k_d) \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}, \quad \Omega = \Omega + \sum_{k=1}^{d} k_i e_i, \end{cases} \tag{8}$$

where $(e_i)_{1\leqslant i\leqslant N}$ is the canonical basis of $\mathbb{R}^N$. In other words, Ω is bounded in the y-direction and periodic in x. As archetypes of the domain Ω, we may have the whole space $\mathbb{R}^N$ which corresponds for $d = N$ and $L_1,\dots,L_N$ any array of positive real numbers. We may also have the whole space $\mathbb{R}^N$ with a periodic array of holes or an infinite cylinder with an oscillating boundary. In this periodic situation, we call C the periodicity cell of Ω,

$$C = \big\{(x,y) \in \Omega;\ x_1 \in (0,L_1),\dots,x_d \in (0,L_d)\big\}. \tag{9}$$

The diffusion matrix $A(x,y) = (A_{ij}(x,y))_{1\leqslant i,j\leqslant N}$ is a *symmetric* $C^{2,\delta}(\overline{\Omega})$ (with $\delta > 0$) matrix field which is L-periodic with respect to x and satisfies:

$$\exists 0 < \alpha_1 \leqslant \alpha_2, \forall (x,y) \in \Omega,\ \forall \xi \in \mathbb{R}^N, \quad \text{we have } \alpha_1|\xi|^2 \leqslant \sum_{1\leqslant i,j\leqslant N} A_{ij}(x,y)\xi_i\xi_j \leqslant \alpha_2|\xi|^2. \tag{10}$$

The underlying advection $q(x,y) = (q_1(x,y),\dots,q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) vector field which is L-periodic with respect to x and satisfies:

$$\nabla\cdot q = 0 \quad \text{in } \overline{\Omega},\ q\cdot\nu = 0 \text{ on } \partial\Omega \text{ (when } \partial\Omega \neq \emptyset),\ \forall 1 \leqslant i \leqslant d, \int_C q_i \,\mathrm{d}x\,\mathrm{d}y = 0. \tag{11}$$

Concerning the nonlinearity $f = f(x,y,u)$, it is a nonnegative function defined in $\overline{\Omega} \times [0,1]$, such that

$$\begin{cases} f \text{ is } L\text{-periodic with respect to } x, \text{ and of class } C^{1,\delta}\big(\overline{\Omega}\times[0,1]\big), \\ \forall (x,y) \in \overline{\Omega}, \quad f(x,y,0) = f(x,y,1) = 0, \\ \exists \rho \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall 1-\rho \leqslant s \leqslant s' \leqslant 1,\ f(x,y,s) \geqslant f(x,y,s'), \\ \forall (x,y) \in \overline{\Omega}, \quad \zeta(x,y) := f'_u(x,y,0) = \lim_{u\to 0^+} \dfrac{f(x,y,u)}{u} > 0, \end{cases} \tag{12}$$

with the additional "KPP" assumption (referring to [8] by Kolmogorov, Petrovsky and Piskunov)

$$\forall (x,y,s) \in \overline{\Omega} \times (0,1), \quad 0 < f(x,y,s) \leqslant f'_u(x,y,0) \times s. \tag{13}$$

An archetype of f is $(x,y,u) \mapsto u(1-u)h(x,y)$ defined on $\overline{\Omega} \times [0,1]$ where h is a positive $C^{1,\delta}(\overline{\Omega})$ L-periodic function.

In all of this paper, $e \in \mathbb{R}^d$ is a fixed unit vector and $\tilde{e} := (e,0,\dots,0) \in \mathbb{R}^N$. A pulsating travelling front propagating in the direction of $-e$ within a speed $c \neq 0$ is a solution $u = u(t,x,y)$ of (7) for which there exists a function ϕ such that $u(t,x,y) = \phi(x\cdot e + ct, x, y)$, ϕ is L-periodic in x and

$$\lim_{s\to-\infty} \phi(s,x,y) = 0 \quad \text{and} \quad \lim_{s\to+\infty} \phi(s,x,y) = 1,$$

uniformly in $(x,y) \in \overline{\Omega}$.

In the same setting as in this paper, it was proved in [1] and [3] that for all Ω, A, q, and f satisfying (8), (10), (11), and (12) respectively, there exists $c^*_{\Omega,A,q,f}(e)$, called the *minimal speed of propagation*, such that pulsating travelling fronts exist if and only if $c \geqslant c^*_{\Omega,A,q,f}(e)$. This result extended that of [8] which proved that $c^*(e) = 2\sqrt{f'(0)}$ in a "homogeneous" framework where $f = f(u)$ and there is no advection q. A variational formula for the minimal speed $c^*_{\Omega,A,q,f}(e)$ involving the principal eigenvalue of an elliptic operator was proved in [3]. Moreover, El Smaily [6] proved a min–max formula for the minimal speed. In the following, we recall the definition of "first integrals" of a vector field which was introduced in [2].

Definition 1.1 *(First integrals).* The family of first integrals of an incompressible advection q of the type (11) is defined by

$$\mathcal{I} := \big\{w \in H^1_{loc}(\overline{\Omega}),\ w \neq 0,\ w \text{ is } L\text{-periodic in } x, \text{ and } q\cdot\nabla w = 0 \text{ almost everywhere in } \Omega\big\}.$$

Having a matrix $A = A(x,y)$ of the type (10), we also define:

$$\mathcal{I}_1^A := \left\{w \in \mathcal{I},\ \text{such that} \int_C \zeta w^2 \geqslant \int_C \nabla w\cdot A\nabla w\right\}. \tag{14}$$

The following theorem gives the asymptotic behavior of the minimal speed in the presence of a large advection in any dimension N.

Theorem 1.2. *We fix a unit direction $e \in \mathbb{R}^d$ and assume that the diffusion matrix A and the nonlinearity f satisfy* (10), (12) *and* (13). *Let q be an advection field which satisfies* (11). *Then,*

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} = \max_{w\in\mathcal{I}_1^A} \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}. \tag{15}$$

The above theorem was proved in details in [7] and [9]. In [9], Zlatoš treated the problem when the domain Ω is the whole $\mathbb{R}^N$. Other asymptotics of the minimal speed were proved in El Smaily [4] and El Smaily, Hamel, Roques [5].

In the case where $N = 2$, we give necessary and sufficient conditions on the streamlines (or the trajectories) of the advection field q for which the limit (15) is positive or null. In the following lemma, we describe the family of "unbounded periodic trajectories" of L-periodic 2-dimensional vector fields q.

Lemma 1.3 *(Unbounded periodic trajectories). We assume here that $N = 2$ and hence $d \in \{1,2\}$. Assume that q satisfies* (11). *Let $T(x)$ be an unbounded periodic trajectory of q in Ω. That is, there exists $\mathbf{a} \in L_1\mathbb{Z}\times L_2\mathbb{Z}\setminus\{0\}$ (resp. $L_1\mathbb{Z}\times\{0\}\setminus\{0\}$) when $d = 2$ (resp. $d = 1$) such that $T(x) = T(x) + \mathbf{a}$. In this case, we say that $T(x)$ is $\mathbf{a}$-periodic. Then, if $T(y)$ is another unbounded periodic trajectory of q, $T(y)$ is also $\mathbf{a}$-periodic.*

Moreover, in the case $d = 1$, $\mathbf{a} = L_1e_1$. That is, all the unbounded periodic trajectories of q in Ω are L_1e_1-periodic.

Theorem 1.4. *Assume that $N = 2$ and that Ω and q satisfy* (8) *and* (11) *respectively. The two following statements are equivalent:*

(i) *There exists $w \in \mathcal{I}$, such that $\int_C qw^2 \neq 0$.*
(ii) *There exists a periodic unbounded trajectory $T(x)$ of q in Ω.*

Moreover, if (ii) *is verified and $T(x)$ is $\mathbf{a}$-periodic, then for any $w \in \mathcal{I}$ we have $\int_C qw^2 \in \mathbb{R}\mathbf{a}$.*

Remark 1. The periodicity assumption on the trajectory in (ii) is crucial. Indeed there may exist unbounded trajectories which are not periodic, even though the vector field q is periodic. Consider, for example,the following function ϕ:

$$\phi(x,y) := \begin{cases} e^{-\frac{1}{\sin^2(\pi y)}} \sin\bigl(2\pi\bigl(x + \ln\bigl(y - [y]\bigr)\bigr)\bigr) & \text{if } y \notin \mathbb{Z}, \\ 0 \quad \text{otherwise}, \end{cases}$$

where $[y]$ denotes the integer part of y. This function is C^∞ on $\mathbb{R}^2$, and 1-periodic in x and y. Hence the vector field

$$q = \nabla^\perp\phi$$

is also C^∞, 1-periodic in x and y, and verifies $\int_{[0,1]\times[0,1]} q = 0$ with $\nabla\cdot q \equiv 0$. A quick study of this vector field shows that the part of the graph of $x \mapsto e^{-x}$ lying between $y = 0$ and $y = 1$ is a trajectory of q, and is obviously unbounded and not periodic. Moreover, there exist no periodic unbounded trajectories for this vector field, so the theorem asserts that for all $w \in \mathcal{I}$ we have:

$$\int_C qw^2 = 0.$$

Corollary 1.5. *Assume that $N = 2$ and that Ω, A, q and f satisfy the conditions* (8), (10), (11) *and* (12)–(13) *respectively. Then,*

(i) *If there exists no periodic unbounded trajectory of q in Ω, and*

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} = 0,$$

for any unit direction e.

(ii) *If there exists a periodic unbounded trajectory $T(x)$ of q in Ω (which will be $\mathbf{a}$-periodic for some vector $\mathbf{a} \in \mathbb{R}^2$) and*

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} > 0 \iff \tilde{e}\cdot\mathbf{a} \neq 0. \tag{16}$$

We mention that in the case where $d = 1$, we have $\tilde{e} = \pm e_1$. Lemma 1.3 yields that $\tilde{e}\cdot\mathbf{a} = \pm L_1 \neq 0$. Referring to (16), we can then write, for $d = 1$,

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} > 0 \iff \bigl(\textit{there exists a periodic unbounded trajectory } T(x) \textit{ of } q \textit{ in } \Omega\bigr). \tag{17}$$

In order to prove Theorem 1.4, we proved the following qualitative property for the family of the advection fields that we consider in this work. The following proposition can be viewed as a generalization of the Hodge representation to unbounded domains with a periodic structure.

Proposition 1.6. *Let $N = 2$, $d = 1$ or 2 where d is defined in* (8)*. Let $q \in C^{1,\delta}(\overline{\Omega})$, L-periodic with respect to x and verifying the conditions* (11)*. Then, there exists $\phi \in C^{2,\delta}(\overline{\Omega})$, L-periodic with respect to x, such that*

$$q = \nabla^{\perp}\phi \quad \text{in } \Omega. \tag{18}$$

Moreover, ϕ is constant on every connected component of $\partial\Omega$.

Remark 2. We mention that the representation $q = \nabla^{\perp}\phi$ is already known in the case where the domain Ω is bounded and *simply connected* or equal to whole space $\mathbb{R}^2$. However, the above proposition applies in more cases due to the condition $q \cdot \nu = 0$ on $\partial\Omega$. For example, it applies when Ω is the whole space $\mathbb{R}^2$ with a periodic array of holes or when Ω is an infinite cylinder which may have an oscillating boundary and/or a periodic array of holes.

The proofs of all the above results, further details and more asymptotic properties of the KPP minimal speed are shown in [7].

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