C. R. Acad. Sci. Paris, Ser. I 348 (2010) 743–746

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical Analysis

Decomposition of $\mathbb{S}^1$-valued maps in Sobolev spaces

Décomposition des applications unimodulaires dans les espaces de Sobolev

Petru Mironescu

Université de Lyon, CNRS, Université Lyon 1, Institut Camille-Jordan, 43, boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France

ARTICLE INFO

Article history:
Received 22 June 2010
Accepted 23 June 2010
Available online 10 July 2010

Presented by Haïm Brezis

ABSTRACT

Let $n \geqslant 2$, $s > 0$, $p \geqslant 1$ be such that $1 \leqslant sp < 2$. We prove that for each map $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ one can find $\varphi \in W^{s,p}(\mathbb{S}^n;\mathbb{R})$ and $v \in W^{sp,1}(\mathbb{S}^n;\mathbb{S}^1)$ such that $u = ve^{\imath\varphi}$. This yields a decomposition of u into a part that has a lifting in $W^{s,p}$, $e^{\imath\varphi}$, and a map "smoother" than u but without lifting, namely v. Our result generalizes a previous one of Bourgain and Brezis (which corresponds to the case $s = 1/2$, $p = 2$). As a consequence, we find an intuitive proof for the existence of the distributional Jacobian Ju of maps $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ (originally due to Bourgain, Brezis and the author). By completing a result of Bousquet, we characterize the distributions of the form Ju.

RÉSUMÉ

Soient $n \geqslant 2$, $s > 0$, $p \geqslant 1$ tels que $1 \leqslant sp < 2$. Nous montrons que, pour chaque $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$, il existe $\varphi \in W^{s,p}(\mathbb{S}^n;\mathbb{R})$ et $v \in W^{sp,1}(\mathbb{S}^n;\mathbb{S}^1)$ tels que $u = ve^{\imath\varphi}$. Ceci donne une décomposition de u comme produit d'un facteur qui se relève dans $W^{s,p}$, $e^{\imath\varphi}$, et d'un facteur « plus régulier » que u mais qui ne se relève pas, à savoir v. Notre décomposition généralise un résultat antérieur de Bourgain et Brezis (qui ont traité le cas $s = 1/2$, $p = 2$). Une conséquence de notre résultat est une preuve intuitive de l'existence du jacobien au sens des distributions Ju pour les applications $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ (résultat dû, avec un argument différent, à Bourgain, Brezis et l'auteur). En complétant un résultat de Bousquet, nous caractérisons les distributions de la forme Ju.

1. Decomposition of $\mathbb{S}^1$-valued maps

Our main result is the following:

Theorem 1. *Let $n \geqslant 2$, $s > 0$, $p \geqslant 1$ be such that $1 \leqslant sp < 2$. Let $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$. Then there exist $\varphi \in W^{s,p}(\mathbb{S}^n;\mathbb{R})$ and $v \in W^{sp,1}(\mathbb{S}^n;\mathbb{S}^1)$ such that $u = ve^{\imath\varphi}$.*

In addition, we have (with $|\cdot|_{W^{r,q}}$ standing for the semi-norm given by the highest order term in $\|\cdot\|_{W^{r,q}}$)

$$|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}}, \ |v|_{W^{sp,1}} \lesssim |u|^p_{W^{s,p}}. \tag{1}$$

E-mail address: mironescu@math.univ-lyon1.fr.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.06.020

The special case $s = 1/2$, $p = 2$ of Theorem 1 is due to Bourgain and Brezis [4]. (In [4], u is supposed to be in the $H^{1/2}$-closure of $C^\infty(\mathbb{S}^n;\mathbb{S}^1)$. This extra assumption was removed in [6].) In Theorem 1, $\mathbb{S}^n$ does not play special role; one could replace, e.g., $\mathbb{S}^n$ by any smooth bounded simply connected domain. Theorem 1 yields a satisfactory substitute to the lifting theory in $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$, theory developed successively in [5,18] and [14]. As proved in these papers, when $n \geqslant 2$ and $sp \notin [1,2)$, one may characterize maps $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ in terms of their liftings. (For a precise statement, we refer to [15], Theorem 6.1, p. 15.) However, when $1 \leqslant sp < 2$, there is no satisfactory description of maps in terms of their phases. A typical example is the map $\mathbb{C} \ni z \mapsto z/|z|$, which belongs to $W^{s,p}(B(0,1))$ when $sp < 2$, but does not have a phase better than $z \mapsto \arg z$, which merely belongs to BV. Our result allows to decompose u into two parts, one as smooth as u and which admits a lifting in $W^{s,p}$, the other one without lifting in $W^{s,p}$, but "smoother" than u. In Theorem 1, one cannot replace $W^{s,p}$ (for φ) or $W^{sp,1}$ (for v) by smaller Sobolev spaces.

The proof of Theorem 1 is constructive: there is an explicit formula giving φ. Part of the proof is inspired by similar constructions of Bourgain and Brezis [4] and of the author [14]. We describe the main lines of the proof when $s < 1$ and $1 \leqslant sp < 2$, and when $\mathbb{S}^n$ is replaced by B, the unit ball in $\mathbb{R}^n$. We extend $u \in W^{s,p}(B;\mathbb{S}^1)$ to $\mathbb{R}^n$ by reflections and cutoff. We let $\Pi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that $\Pi(z) = z/|z|$ when $|z| \geqslant 1/2$ and let ρ be a suitable mollifier. With $w(x,\varepsilon) := u * \rho_\varepsilon(x)$, $x \in \mathbb{R}^n$, $\varepsilon > 0$, we set, inspired by [14],

$$\varphi_1(x) := -\int_0^\infty \Pi \circ w(x,\varepsilon) \wedge \frac{\partial}{\partial \varepsilon}(\Pi \circ w)(x,\varepsilon)\,\mathrm{d}\varepsilon.$$

This φ_1 satisfies $\varphi_1 \in W^{s,p}(B)$ and $U := ue^{-\imath\varphi_1} \in W^{1,sp}(B)$. If $sp = 1$, then we may take $\varphi = \varphi_1$. When $1 < sp < 2$, two more steps are needed. We extend U to $\mathbb{R}^n$ by reflections and cutoff and define $\varphi_2 := \sum_k \sum_{j<k} U_j \wedge U_k$. Here, $U = \sum U_j$ is a Littlewood–Paley decomposition of U. The idea of improving the regularity of a map with the help of this phase originates in the paper [4] of Bourgain and Brezis. This φ_2 satisfies $\varphi_2 \in W^{1,sp}$ and $Ue^{-\imath\varphi_2} \in W^{sp,1}(B)$.

Third step: since $\varphi_2 \in W^{1,sp}$, we have $\varphi_2 = \varphi_3 + \varphi_4$, where $\varphi_3 \in W^{s,p}$ and $\varphi_4 \in W^{sp,1} \cap W^{1,sp}$. The regularity of φ_4 implies that $e^{\imath\varphi_4} \in W^{sp,1}$ [10,13]. Thus $u = e^{\imath\varphi} v$, where $\varphi := \varphi_1 + \varphi_3 \in W^{s,p}$ and $v := Ue^{\imath\varphi_4} \in W^{sp,1}$.

2. The distributional Jacobian revisited

We recall the definition of the distributional Jacobian for $\mathbb{S}^1$-valued maps [17,19,2,9,12,1,6,7]. If $u = (u_1,u_2) \in W^{1,1}(\mathbb{S}^2;\mathbb{S}^1)$, then $Ju := \frac{1}{2}\,\mathrm{d}(u_1\,\mathrm{d}u_2 - u_2\,\mathrm{d}u_1)$. This distribution (current) coincides with the usual Jacobian 2-form $\mathrm{d}u_1 \wedge \mathrm{d}u_2$ if u is sufficiently smooth, say $u \in H^1$. In the latter case, $Ju = 0$ for $\mathbb{S}^1$-valued maps u. As a distribution, Ju is defined by

$$\langle Ju, \zeta\rangle = \frac{1}{2}\int_{\mathbb{S}^2} (u_1\,\mathrm{d}u_2 - u_2\,\mathrm{d}u_1) \wedge \mathrm{d}\zeta, \quad \forall \zeta \in C^\infty\bigl(\mathbb{S}^2;\mathbb{R}\bigr). \tag{2}$$

More generally, when $u \in W^{1,1}(\mathbb{S}^n;\mathbb{S}^1)$, Ju is defined as an $(n-2)$-current through the formula

$$\langle Ju, \zeta\rangle = \frac{1}{2}\int_{\mathbb{S}^n} (u_1\,\mathrm{d}u_2 - u_2\,\mathrm{d}u_1) \wedge \mathrm{d}\zeta, \quad \forall \zeta \in \Lambda^{n-2}\bigl(\mathbb{S}^n\bigr). \tag{3}$$

The following result was proved in [6]:

Theorem 2. *(See [6].) Let $n \geqslant 2$, $s > 0$, $p \geqslant 1$ be such that $1 \leqslant sp < 2$. Then $W^{s,p} \cap W^{1,1}(\mathbb{S}^n;\mathbb{S}^1)$ is dense in $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$. In addition, the map $u \mapsto Ju$ extends by continuity from $W^{s,p} \cap W^{1,1}(\mathbb{S}^n;\mathbb{S}^1)$ to $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$.*

Denoting by $u \mapsto Ju$ this extension, Theorem 1 sheds a new light on Theorem 2 via the following:

Proposition 3. *Let $n \geqslant 2$, $s > 0$, $p \geqslant 1$ be such that $1 \leqslant sp < 2$. Let $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ and write $u = ve^{\imath\varphi}$, with $\varphi \in W^{s,p}$ and $v \in W^{sp,1}$. Then, for each choice of φ and v, we have*

$$\langle Ju, \zeta\rangle = \frac{1}{2}\int_{\mathbb{S}^n} (v_1\,\mathrm{d}v_2 - v_2\,\mathrm{d}v_1) \wedge \mathrm{d}\zeta, \quad \forall \zeta \in \Lambda^{n-2}\bigl(\mathbb{S}^n\bigr). \tag{4}$$

3. Existence of maps with prescribed singularities. The two dimensional case

Set $\mathcal{R} := \{u \in W^{1,1}(\mathbb{S}^2;\mathbb{S}^1);\, u$ is smooth outside some finite set $A = A(u)\}$. When $u \in \mathcal{R}$, we have $\langle Ju,\zeta\rangle = \pi\sum_{a\in A} d_a\zeta(a)$, where the integers d_a are the degrees of u on suitably oriented small circles around $a \in A$ and satisfy $\sum d_a = 0$ [12]. Thus $Ju = \pi\sum_{a\in A} d_a\delta_a$. Since $\mathcal{R}$ is dense in $W^{1,1}(\mathbb{S}^2;\mathbb{S}^1)$ [3], one obtains that $\{Ju;\, u \in W^{1,1}(\mathbb{S}^2;\mathbb{S}^1)\} \subset E_{1,1}$, where

$$E_{1,1} := \pi\,\overline{\left\{\sum(\delta_{P_j} - \delta_{N_j})\right\}}^{(W^{1,\infty})^*}.$$

The reversed inclusion is true.

Theorem 4. *(See [1,11].) We have* $\{Ju;\ u \in W^{1,1}(\mathbb{S}^2;\mathbb{S}^1)\} = E_{1,1}$.

Bousquet [7] partially completed this result.

Theorem 5. *(See [7].) Assume that* $s \geqslant 1$ *and* $1 \leqslant sp < 2$. *Then* $\{Ju;\ u \in W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)\} = E_{s,p}$, *where*

$$E_{s,p} := \pi\,\overline{\left\{\sum(\delta_{P_j} - \delta_{N_j})\right\}}^{(W^{1,sp/(sp-1)})^* \cap (W^{2-s,p/(p-1)})^*}.$$

Note that the definition of $E_{s,p}$ suggests that different values of s and p yield different $E_{s,p}$'s. Our first result in this direction is somewhat surprising.

Theorem 6. *Assume that* $s \geqslant 1$ *and* $1 \leqslant sp < 2$. *Then* $E_{s,p} = E_{1,sp}$.

In particular, if a (possible infinite) sum of the form $\sum(\delta_{P_j} - \delta_{N_j})$, *with* $\sum|P_j - N_j| < \infty$, *acts on* $W^{1,r}$ *for some* $r \in (2,\infty)$, *then it also acts on the Hölder space* $C^{2-r/(r-1)}$.

As a byproduct, the proof of the above theorem yields the following curious estimate:

$$\left\|\sum(\delta_{P_j} - \delta_{N_j})\right\|_{(C^\alpha)^*} \leqslant K_\alpha \left\|\sum(\delta_{P_j} - \delta_{N_j})\right\|^{2-\alpha}_{(W^{1,(2-\alpha)/(1-\alpha)})^*}, \tag{5}$$

with K_α depending on $0 < \alpha < 1$ but independent of the P_j's and N_j's.

Our next result completes Theorem 5.

Theorem 7. *Assume that* $1 \leqslant sp < 2$. *Then* $\{Ju;\ u \in W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)\} = E_{1,sp}$.

4. Existence of maps with prescribed singularities. The higher dimensional case

In dimension 3 or higher, the class $\mathcal{R}$ is defined as

$$\mathcal{R} := \left\{u \in W^{1,1}\left(\mathbb{S}^n;\mathbb{S}^1\right);\ u \text{ is smooth outside some } (n-2)\text{-submanifold without boundary} A = A(u) \text{ of } \mathbb{S}^n\right\}.$$

If $u \in \mathcal{R}$, then we may identify Ju with the $(n-2)$-current $\pi\sum d_j \int_{\Gamma_j}$, where Γ_j are the (orientable, without boundary) connected components of A and the integers d_j are the degrees of u on suitably oriented small circles linking to the Γ_j's [12,1,7]. We then define, for $1 < q < 2$,

$$E_{1,q} := \pi\,\overline{\left\{\sum d_j \int_{\Gamma_j}\right\}}^{(W^{1,q/(q-1)})^*}.$$

For $q = 1$, the suitable higher dimensional analog of $E_{1,1}$ was pointed out by Alberti, Baldo, Orlandi [1] and is given by $E_{1,1} := \pi\ \{\partial M;\ M \text{ is a rectifiable } (n-1)\text{-current}\}$. With these notations, we have

Theorem 8. *Assume that* $n \geqslant 3$ *and* $1 \leqslant sp < 2$. *Then* $\{Ju;\ u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)\} = E_{1,sp}$.

The case $s = 1$, $p = 1$ was known before [1]. The case $sp = 1$ was obtained jointly with Bousquet [8]. The case $1 < sp < 2$ relies on Theorem 1 and on techniques from [7]. Finally, the analog of (5) is given by

$$\left\|\sum d_j \int_{\Gamma_j}\right\|_{(C^\alpha)^*} \leqslant K_\alpha \left\|\sum d_j \int_{\Gamma_j}\right\|^{2-\alpha}_{(W^{1,(2-\alpha)/(1-\alpha)})^*}, \quad 0 < \alpha < 1. \tag{6}$$

Detailed proofs will appear in [16].

Acknowledgement

The author warmly thanks Pierre Bousquet and Haïm Brezis for useful discussions.

References

[1] G. Alberti, S. Baldo, G. Orlandi, Functions with prescribed singularities, J. Eur. Math. Soc. 5 (3) (2003) 275–311.
[2] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (1977) 337–403.
[3] F. Bethuel, X. Zheng, Density of smooth functions between two manifolds in Sobolev spaces, J. Funct. Anal. 80 (1988) 60–75.
[4] J. Bourgain, H. Brezis, On the equation $\operatorname{div} Y = f$ and application to control of phases, J. Amer. Math. Soc. 16 (2003) 393–426.
[5] J. Bourgain, H. Brezis, P. Mironescu, Lifting in Sobolev spaces, J. Anal. Math. 80 (2000) 37–86.
[6] J. Bourgain, H. Brezis, P. Mironescu, $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg–Landau equation, Publ. Math. Inst. Hautes Études Sci. 99 (2004) 1–115.
[7] P. Bousquet, Topological singularities in $W^{s,p}(\mathbb{S}^N, \mathbb{S}^1)$, J. Anal. Math. 102 (2007) 311–346.
[8] P. Bousquet, P. Mironescu, in preparation.
[9] H. Brezis, J.-M. Coron, E. Lieb, Harmonic maps with defects, Comm. Math. Phys. 107 (4) (1986) 649–705.
[10] H. Brezis, P. Mironescu, Gagliardo–Nirenberg, composition and products in fractional Sobolev spaces, J. Evol. Equ. 1 (2001) 387–404.
[11] H. Brezis, P. Mironescu, A. Ponce, $W^{1,1}$-maps with values into $\mathbb{S}^1$, in: S. Chanillo, P.D. Cordaro, N. Hanges, J. Hounie, A. Meziani (Eds.), Geometric Analysis of PDE and Several Complex Variables, in: Contemporary Mathematics, vol. 368, Amer. Math. Soc., Providence, RI, 2005, pp. 69–100.
[12] R.L. Jerrard, H.M. Soner, Functions of bounded higher variation, Indiana Univ. Math. J. 51 (2002) 645–677.
[13] V. Maz'ya, T. Shaposhnikova, An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces, J. Evol. Equ. 2 (1) (2002) 113–125.
[14] P. Mironescu, Lifting default for $\mathbb{S}^1$-valued maps, C. R. Acad. Sci. Paris, Ser. I 346 (19–20) (2008) 1039–1044.
[15] P. Mironescu, $\mathbb{S}^1$-valued Sobolev maps, http://math.univ-lyon1.fr/~mironescu/7.pdf.
[16] P. Mironescu, Sobolev spaces of circle-valued maps, in preparation.
[17] C.B. Morrey, Multiple Integrals in the Calculus of Variations, Die Grundlehren der mathematischen Wissenschaften, vol. 130, Springer-Verlag New York, Inc., New York, 1966.
[18] H.-M. Nguyen, Inequalities related to liftings and applications, C. R. Acad. Sci. Paris, Ser. I 346 (17–18) (2008) 957–962.
[19] Y.G. Reshetnyak, The weak convergence of completely additive vector-valued set functions, Sibirsk. Mat. Zh. 9 (1968) 1386–1394.