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Number Theory

A ring of periods for Sen modules in the imperfect residue field case

Un anneau des périodes pour les modules de Sen dans le cas du corps résiduel imparfait

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ABSTRACT

We construct a ring $\mathbb{B}_{\mathrm{Sen}}$ of Colmez in the imperfect residue field case.

RÉSUMÉ

Nous construisons un anneau $\mathbb{B}_{\mathrm{Sen}}$ de Colmez dans le cas d'un corps résiduel imparfait.

1. Introduction

In this Note, we use the notation of [1]: Let p be a prime number, K be a complete discrete valuation field of mixed characteristic $(0, p)$. Let k_K be the residue field of K and assume $[k_K : k_K^p] = p^h < \infty$. Fix lifts $t_1, \dots, t_h \in K$ of a p-basis of k_K and fix a compatible system $\{\zeta_{p^n}\}$ (resp. $\{t_j^{p^{-n}}\}$) of p-power roots of unity (resp. t_j). Put $K_\infty = K(\mu_{p^\infty}, t_1^{p^{-\infty}}, \dots, t_h^{p^{-\infty}})$ and $\mathbb{C}_p = \widehat{\overline{K}}$. Let $G_K = G_{\overline{K}/K}$, $G_{K_\infty} = G_{\overline{K}/K_\infty}$, $\Gamma = G_{K_\infty/K}$.

Let $\mathfrak{g} = \mathbb{Q}_p \ltimes \mathbb{Q}_p^h$ be the $(h+1)$-dimensional p-adic Lie algebra where $\mathbb{Q}_p$ acts on $\mathbb{Q}_p^h$ by the scalar multiplication. Let $\varphi = (1, \mathbf{0})$, $\mu_j = (0, \mathbf{e}_j) \in \mathfrak{g}$ for $1 \leqslant j \leqslant h$, where $\mathbf{e}_j \in \mathbb{Q}_p^h$ is the jth fundamental vector. Then we have

$$[\varphi, \mu_j] = \mu_j, [\mu_j, \mu_k] = 0 \tag{1}$$

for $1 \leqslant j, k \leqslant h$. Let χ be the cyclotomic character and for $1 \leqslant j \leqslant h$, let $s_j : \mathfrak{g} \to \mathbb{Z}_p(1)$ be the 1-cocycle defined by $g(t_j^{p^{-n}})/t_j^{p^{-n}} = \zeta_{p^n}^{s_j(g)}$ for $n \in \mathbb{N}$.

The isomorphism $\Gamma \cong U \ltimes \mathbb{Z}_p(1)^h; \gamma \mapsto (\chi(\gamma), s_1(\gamma), \dots, s_h(\gamma))$ for some open subgroup $U \leqslant \mathbb{Z}_p^\times$ induces an isomorphism $\mathrm{Lie}(\Gamma) \cong \mathfrak{g}$ of p-adic Lie algebras. In the following, we identify $\mathrm{Lie}(\Gamma) = \mathfrak{g}$ by this isomorphism. Note that the usual logarithm map $\log : \Gamma \to \mathfrak{g}$ satisfies

$$\log(\gamma) = \log\bigl(\chi(\gamma)\bigr)\varphi + s_1(\gamma)\mu_1 + \dots + s_h(\gamma)\mu_h$$

for $\gamma \in \Gamma$.

Recall Brinon's generalization of Sen's theory [1]. For a topological field B with a continuous action of a topological group G, denote by $\mathrm{Rep}_B G$ the category of finite dimensional B-vector spaces with continuous, semi-linear G-action. Then there exist canonical equivalences

$$\mathrm{Rep}_{\mathbb{C}_p} G_K \to \mathrm{Rep}_{\widehat{K}_\infty} \Gamma; V \mapsto V^{G_{K_\infty}}, \qquad \mathrm{Rep}_{\widehat{K}_\infty} \Gamma \to \mathrm{Rep}_{K_\infty} \Gamma; V \mapsto V^{\mathrm{f}},$$

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which are quasi-inverse to $V \mapsto \mathbb{C}_p \otimes_{\widehat{K}_\infty} V$ and $V \mapsto \widehat{K}_\infty \otimes_{K_\infty} V$. For $V \in \mathrm{Rep}_{\mathbb{C}_p} G_K$, define $\mathbb{D}_{\mathrm{Sen}}(V)$ as the differential representation, whose dimension over K_∞ is equal to that of V over $\mathbb{C}_p$, associated to $(V^{G_{K_\infty}})^{\mathrm{f}}$. Note that, for $v \in V$, there exists $\Gamma_v \trianglelefteq_{\mathrm{o}} \Gamma$ such that

$$\gamma(v) = \exp\bigl(\log(\gamma)\bigr)(v) \tag{2}$$

for $\gamma \in \Gamma_v$.

In [2], Colmez defined a ring of periods $\mathbb{B}_{\mathrm{Sen}}$ and reconstructed the functor $\mathbb{D}_{\mathrm{Sen}}$ by using this ring in the case $h = 0$. The aim of this paper is to extend his results to the case $h > 0$.

2. Construction of $\mathbb{B}_{\mathrm{Sen}}$

For $n \in \mathbb{N}$, let $K_n = K(\mu_{p^n}, t_1^{p^{-n}}, \ldots, t_h^{p^{-n}})$ and $G_{K_n} = G_{\overline{K}/K_n}$. We say that an abelian group A has a G_{K_∞}-structure if A has an increasing filtration $\{A_n\}_{n\in\mathbb{N}}$, consisting of subgroups of A, such that $A = \bigcup A_n$ and that A_n has a filtration-compatible G_{K_n}-action. In this case, let $A^{G_{K_\infty}} = \bigcup A_n^{G_{K_n}}$.

We construct a ring $\mathbb{B}_{\mathrm{Sen}}$ with a G_{K_∞}-structure as follows: As a ring, $\mathbb{B}_{\mathrm{Sen}}$ is the ring of formal power series of $(h+1)$-variables $X_0, \ldots, X_h$ with coefficients in $\mathbb{C}_p$ converging on $\{(X_0, \ldots, X_h) \in \mathbb{C}_p^{h+1} \mid |X_0|, \ldots, |X_h| < \varepsilon\}$ for some $\varepsilon \in \mathbb{R}_{>0}$. For $n \in \mathbb{N}$, let $\mathbb{B}_{\mathrm{Sen}}^n$ be the subring consisting of power series converging on the polydisc $\{X = (X_0, \ldots, X_h) \in \mathbb{C}_p^{h+1} \mid |X_0|, \ldots, |X_h| \leqslant |p^n|\}$. Then G_{K_n} acts on $\mathbb{B}_{\mathrm{Sen}}^n$, semi-linearly on the coefficients, by

$$g(X_0) = X_0 + \log\chi(g), \tag{3}$$

$$g(X_j) = \frac{1}{\chi(g)}\bigl(X_j + s_j(g)\bigr) \tag{4}$$

for $1 \leqslant j \leqslant h$. These data give $\mathbb{B}_{\mathrm{Sen}}$ a G_{K_∞}-structure. Finally, let $\partial_0, \ldots, \partial_h \in \mathrm{Der}_{\mathbb{C}_p}^{\mathrm{cont}}(\mathbb{B}_{\mathrm{Sen}})$ be the continuous differential operators of $\mathbb{B}_{\mathrm{Sen}}$ defined by

$$\partial_0 = -\frac{\partial}{\partial X_0}, \tag{5}$$

$$\partial_j = -\frac{1}{\exp(X_0)}\frac{\partial}{\partial X_j} \tag{6}$$

for $1 \leqslant j \leqslant h$. For $V \in \mathrm{Rep}_{\mathbb{C}_p} G_K$, endow $\mathbb{B}_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V$ with the G_{K_∞}-structure induced by that of $\mathbb{B}_{\mathrm{Sen}}$ and the action of G_K on V.

Lemma. *(Cf. [2, Théorème 2(i)].)*

(i) *For all* $n \in \mathbb{N}$, $(\mathbb{B}_{\mathrm{Sen}}^n)^{G_{K_n}} = (\mathrm{Frac}\,\mathbb{B}_{\mathrm{Sen}}^n)^{G_{K_n}} = K_n$ *and* $(\mathbb{B}_{\mathrm{Sen}})^{G_{K_\infty}} = K_\infty$.
(ii) *Let* $\mathfrak{g}_{K_\infty} = K_\infty \otimes_{\mathbb{Q}_p} \mathfrak{g}$. *Then* $(\mathbb{B}_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V)^{G_{K_\infty}}$ *is a* $\mathfrak{g}_{K_\infty}$*-representation and we have* $\dim_{K_\infty}(\mathbb{B}_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V)^{G_{K_\infty}} \leqslant \dim_{\mathbb{C}_p} V$.

Proof. (i) Let $x \in (\mathbb{B}_{\mathrm{Sen}}^n)^{G_{K_n}}$. Since $g(x) = x$ for $g \in G_{K_\infty}$, x has coefficients in $\widehat{K}_\infty$. Let $\Gamma_m = G_{K_\infty/K_m}$, $K_m^j = K(\mu_{p^\infty}, t_1^{p^{-\infty}}, \ldots, t_{j-1}^{p^{-\infty}}, t_j^{p^{-m}}, \ldots, t_h^{p^{-m}})$ and let $t_m : \widehat{K}_\infty \to \widehat{K}_m$ be the continuous map characterized by $t_m(x) = \varinjlim_{l \geqslant m}[K_l : K_m]^{-1}\,\mathrm{Tr}_{K_l/K_m}(x)$ for $x \in K_\infty$. (The continuity of the trace follows from the decomposition into the composition of normalized trace maps

$$K_\infty \to K_m^h \to \cdots \to K_m^1 \to K_m,$$

which are continuous by [1, §2, after Lemme 3].) For $g \in \Gamma_m$ with $m \geqslant n$, by substituting $X = \mathbf{0}$ in $g(x) = x$ and taking g^{-1} of both sides, we have $x(\mathbf{a}_m(g)) = g^{-1}(x(\mathbf{0}))$, where $\mathbf{a}_m : \Gamma_m \to \mathbb{Z}_p^{h+1}$; $g \mapsto (\log\chi(g), s_1(g)/\chi(g), \ldots, s_h(g)/\chi(g))$. By taking the trace of both sides, we have $t_m(x)(\mathbf{a}_m(g)) = t_m(x)(\mathbf{0})$, hence $t_m(x)$ is a constant since the image of $\mathbf{a}_m$ contains a polydisc. Note that for $a \in \widehat{K}_\infty$, $t_m(a) = 0$ for all sufficiently large m implies that $a = 0$ by approximating a by a sequence in $\{K_l\}_{l \geqslant m}$. Therefore x is a constant, that is, $x \in \mathbb{C}_p^{G_{K_n}} = K_n$.

Let $z = x/y \in (\mathrm{Frac}\,\mathbb{B}_{\mathrm{Sen}}^n)^{G_{K_n}} \setminus \{0\}$ with $x, y \in \mathbb{B}_{\mathrm{Sen}}^n \setminus \{0\}$. We have only to prove $y \in (\mathbb{B}_{\mathrm{Sen}}^m)^\times$ for sufficiently large m: This implies that $z \in (\mathbb{B}_{\mathrm{Sen}}^m)^{G_{K_m}} = K_m$ by (i) and then $z \in K_m^{G_{K_m/K_n}} = K_n$.

We may assume that x, y are prime to each other. (Note that $\mathbb{B}_{\mathrm{Sen}}^n$ is isomorphic to a Tate algebra, in particular, it is a UFD.) For $g \in G_{K_n}$, we have $g(x)/g(y) = x/y$. Hence we have $g(y) = \eta_g y$ with $\eta_g \in (\mathbb{B}_{\mathrm{Sen}}^n)^\times$.

Now suppose $y(\mathbf{0}) = 0$. Then, just as in the above argument, by substituting $X = \mathbf{0}$ in $g(y) = \eta_g y$ and taking g^{-1} of both sides, we see that y vanishes on some polydisc in $\mathbb{Z}_p^{h+1}$. This implies that $y = 0$, which is a contradiction. Hence we have $y(\mathbf{0}) \neq 0$ and it is easy to see that $y \in (\mathbb{B}_{\mathrm{Sen}}^m)^\times$ for sufficiently large m.

(ii) Since we have $[\partial_0, \partial_j] = \partial_j$, $[\partial_j, \partial_k] = 0$ for $1 \leqslant j, k \leqslant h$ by (5), (6), and these operators commute with the action of G_{K_n} on $\mathbb{B}^n_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V$, the first assertion follows. The latter assertion follows from the injectivity of the comparison map

$$\alpha_n(V) : \mathbb{B}^n_{\mathrm{Sen}} \otimes_{K_n} \big(\mathbb{B}^n_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V\big)^{G_{K_n}} \to \mathbb{B}^n_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V.$$

Suppose that $\alpha_n(V)$ is not injective. Let d be the smallest integer such that there exist linearly independent elements $\mathbf{e}_1, \ldots, \mathbf{e}_d \in (\mathbb{B}^n_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V)^{G_{K_n}} \subset \mathbb{B}^n_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V$ over K_n, which are linearly dependent over $\mathbb{B}^n_{\mathrm{Sen}}$. Choose a non-trivial relation $\sum_i \lambda_i \mathbf{e}_i = 0$ with $\lambda_i \in \mathbb{B}^n_{\mathrm{Sen}} \setminus \{0\}$. Then, by assumption, $g(\lambda_i/\lambda_1) = \lambda_i/\lambda_1$ for all $g \in G_{K_n}$. Hence $\lambda_i/\lambda_1 \in K_n$ by (i), which is a contradiction with the linear independence of the $\mathbf{e}_i$'s over K_n. □

Theorem. *(Cf. [2, Théorème 2(ii)].) There exists a functorial isomorphism* $\mathbb{D}_{\mathrm{Sen}}(V) \to (\mathbb{B}_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V)^{G_{K_\infty}}$ *of finite dimensional* $\mathfrak{g}_{K_\infty}$*-representations.*

Proof. Let $f : \mathbb{D}_{\mathrm{Sen}}(V) \to \mathbb{B}_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V$ be the injective K_∞-linear map defined by

$$\begin{aligned} f(v) &= \exp(-X_1\mu_1 - \cdots - X_h\mu_h)\exp(-X_0\varphi)(v) \\ &= \sum_{(n_0,\ldots,n_h)\in\mathbb{N}^{h+1}} (-1)^{n_0+\cdots+n_h} \frac{X_0^{n_0}X_1^{n_1}\cdots X_h^{n_h}}{n_0!\cdots n_h!} \otimes \mu_1^{n_1}\cdots\mu_h^{n_h}\varphi^{n_0}(v). \end{aligned}$$

We will prove that this induces the desired isomorphism. Since we have $f \circ \varphi = \partial_0 \circ f$, $f \circ \mu_j = \partial_j \circ f$ for $1 \leqslant j \leqslant h$ by (1), (5), (6), f is a morphism of $\mathfrak{g}_{K_\infty}$-representations. To prove that f is an isomorphism, since we have $\dim_{K_\infty}(\mathbb{B}_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V)^{G_{K_\infty}} \leqslant \dim_{\mathbb{C}_p} V = \dim_{K_\infty} \mathbb{D}_{\mathrm{Sen}}(V)$ by Lemma (ii), we have only to prove that, for $v \in \mathbb{D}_{\mathrm{Sen}}(V)$, we have $f(v) \in (\mathbb{B}^n_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V)^{G_{K_n}}$ for sufficiently large n.

Recall that we have relations

$$g \circ \varphi = \big(\varphi - s_1(g)\mu_1 - \cdots - s_h(g)\mu_h\big) \circ g, \tag{7}$$

$$g \circ \mu_j = \chi(g)\mu_j \circ g \tag{8}$$

for $g \in \Gamma$ and $1 \leqslant j \leqslant h$. (The proof is similar to that of [1, Proposition 7].)

Obviously, the action of G_K on $\mathrm{Im}(f)$ factors through Γ. Let $\Gamma^0 = G_{K_\infty/K(t_1^{p^{-\infty}},\ldots,t_h^{p^{-\infty}})}$ and $\Gamma^j = G_{K_\infty/K(\mu_{p^\infty}, t_1^{p^{-\infty}},\ldots,t_{j-1}^{p^{-\infty}}, t_{j+1}^{p^{-\infty}}, t_h^{p^{-\infty}})}$ for $1 \leqslant j \leqslant h$. These subgroups topologically generate Γ. In the following, we prove that for $v \in \mathbb{D}_{\mathrm{Sen}}(V)$, for $0 \leqslant j \leqslant h$ and $\gamma \in \Gamma^j$ sufficiently close to 1, one has $\gamma(f(v)) = f(v)$.

For $\gamma \in \Gamma^0 \cap \Gamma_v$, we have

$$\begin{aligned} \gamma\big(f(v)\big) &= \exp\left(-\frac{1}{\chi(\gamma)}X_1 \cdot \chi(\gamma)\mu_1 - \cdots - \frac{1}{\chi(\gamma)}X_h \cdot \chi(\gamma)\mu_h\right)\gamma\big(\exp(-X_0\varphi)(v)\big) \quad \big(\text{by (4), (8)}\big) \\ &= \exp(-X_1\mu_1 - \cdots - X_h\mu_h)\exp\big(-\big(X_0 + \log\chi(\gamma)\big)\varphi\big)\gamma(v) \quad \big(\text{by (3), (7)}\big) \\ &= \exp(-X_1\mu_1 - \cdots - X_h\mu_h)\exp\big(-\big(X_0 + \log\chi(\gamma)\big)\varphi\big)\exp\big(\log\chi(\gamma)\varphi\big)(v) \quad \big(\text{by (2)}\big) \\ &= f(v). \end{aligned}$$

For $\gamma \in \Gamma^j \cap \Gamma_v$, $1 \leqslant j \leqslant h$, we have

$$\begin{aligned} \gamma\big(f(v)\big) &= \exp\big(-X_1\mu_1 - \cdots - \big(X_j + s_j(\gamma)\big)\mu_j - \cdots - X_h\mu_h\big)\gamma\big(\exp(-X_0\varphi)(v)\big) \quad \big(\text{by (4), (8)}\big) \\ &= \exp(-X_1\mu_1 - \cdots - X_h\mu_h)\exp\big(-s_j(\gamma)\mu_j\big)\exp\big(-X_0\big(\varphi - s_j(\gamma)\mu_j\big)\big)\gamma(v) \quad \big(\text{by (1), (3), (7)}\big) \\ &= \exp(-X_1\mu_1 - \cdots - X_h\mu_h)\exp\big(-s_j(\gamma)\mu_j\big)\exp\big(-X_0\big(\varphi - s_j(\gamma)\mu_j\big)\big)\exp\big(s_j(\gamma)\mu_j\big)(v) \quad \big(\text{by (2)}\big) \\ &= \exp(-X_1\mu_1 - \cdots - X_h\mu_h)\exp\big(-s_j(\gamma)\mu_j\big)\exp\big(s_j(\gamma)\mu_j\big)\exp(-X_0\varphi)(v) \quad \big(\text{by (1)}\big) \\ &= f(v). \qquad \square \end{aligned}$$

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