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Continuous orbit transitions in a one-dimensional inelastic particle system

Transitions continues entre orbites dans un système de particules inélastique unidimensionnel

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ABSTRACT

Continuous transitions between different periodic orbits in a one-dimensional inelastic particle system with two particles are investigated. We explain why continuous transitions that occur when adding or subtracting a single collision are, generically, of co-dimension 2. However, we show that there are an infinite set of degenerate transitions of co-dimension 1. We provide an analysis that gives a simple criteria to classify which transitions are degenerated purely from the discrete set of collisions that occur in the orbits.

RÉSUMÉ

Nous étudions les transitions continues entre différentes orbites périodiques dans un système unidimensionnel inélastique à deux particules. Nous expliquons pourquoi les transitions continues qui apparaissent lorsque l'on ajoute ou enlève une collision sont, en général, de codimension 2. Cependant, nous montrons qu'il existe un ensemble infini de transitions dégénérées de codimension 1. Nous fournissons une méthode qui, en se basant uniquement sur l'ensemble des collisions qui interviennent dans les orbites, donne un critère simple pour déterminer quelles transitions sont dégénérées.

1. Introduction

There are a broad range of industrial applications in which particles with different masses are constrained to move on a line and interact dissipatively. The periodic behavior of such systems is of particular interest because simple periodic dynamics can avoid chattering behavior that can result in excessive machine wear. Systems with particles of equal mass have been studied extensively [1,3,6]. In particular, there can be non-unique periodic orbits [6]. When the masses of the particles are different, large numbers of periodic orbits can coalesce onto a single orbit at a critical parameter value [4,5].

In this paper, we will study the continuous orbit transitions between different periodic orbits. Generically, we show that continuous orbit transitions when adding or subtracting a single collision are of co-dimension 2. Surprisingly, we also show

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that there are an infinite set of degenerate transitions of co-dimension 1. We provide a theoretical analysis that can explain the nature of the degeneracy. Moreover, we show how one can classify continuous orbit transitions as being degenerate or nondegenerate purely using the sequences of collisions in the respective orbits.

2. Formulation

We consider a system of two perfectly rigid particles, whose motion is constrained on a line between two walls. We assume that the right wall is fixed and the left wall is vibrating so that energy is added when particles collide with it. For simplicity we adopt a "sawtooth" motion [2] for the left wall, in which the wall moves with a constant speed V over a distance d before executing an instantaneous jump back to its starting location. Furthermore, the distance d is assumed to be much smaller than the distance between two walls, and thus, we assume all collisions with the left vibrating wall occur at the same location. We only consider point particles, since the physical size of the particles does not affect the motion. We choose scales such that the distance between the two walls, the mass of the left particle and the speed of the left vibrating wall are all unity. We define the mass of the right particle to be m. In the absence of collisions, the velocities of the two particles are constant. The collisions between particles and walls are assumed to be elastic. When the two particles collide, they do so inelastically with coefficient of the restitution e [4]. The total momentum of the two particles is conserved and the velocities of the left and right particles, V_1 and V_2, are updated using the operator C. Similarly, we use the operator L to denote collisions between the left particle and the left wall and R to denote collisions between the right particle and the right wall. The operators C, L and R are defined as

$$C\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{1+m}\begin{bmatrix} 1-em & m(1+e) \\ 1+e & m-e \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \qquad L\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2-V_1 \\ V_2 \end{bmatrix} \quad \text{and} \quad R\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}.$$

We define $V^{(k)} = [V_1^{(k)}, V_2^{(k)}]^T$ to be the velocities of the two particles after the k-th collision and $X^{(k)} = [X_1^{(k)}, X_2^{(k)}]^T$ to be the locations of the two particles when the k-th collision occurs. We denote the initial velocity and location vectors as $V^{(0)}$ and $X^{(0)}$, respectively. We then can compute

$$t_*^{(k)} = \min\bigl\{\bigl(X_j^{(k)} - X_{j+1}^{(k)}\bigr)/\bigl(V_{j+1}^{(k)} - V_j^{(k)}\bigr) \bigm| \bigl(X_j^{(k)} - X_{j+1}^{(k)}\bigr)/\bigl(V_{j+1}^{(k)} - V_j^{(k)}\bigr) > 0,\ j = 0, 1, 2\bigr\},$$

where $X_0^{(k)} \equiv 0$, $X_{N+1}^{(k)} \equiv 1$, $V_0^{(k)} \equiv 0$ and $V_{N+1}^{(k)} \equiv 0$. This $t_*^{(k)}$ is the time at which the next collision occurs. We therefore update the locations using $X^{(k+1)} = X^{(k)} + V^{(k)} t_*^{(k)}$ and velocities $V^{(k)}$ using the corresponding collisions. For given initial velocities and locations, this procedure can be repeated to obtain the times between collisions, the order of collisions, the velocities and locations of the collisions.

In this paper, we will focus on periodic orbits. For example, we consider a given periodic sequence G, that contains K collisions. The periodicity of the velocities requires $V^{(K)} = V^{(0)}$. Since the operators L, C and R can be written in terms of matrices, we can rewrite it as $(I - G')V^{(0)} = G''[2,0]^T$, where I is the identity matrix. G' and G'' are matrices which depend only on the order of the collisions. After solving for $V^{(0)}$, we can calculate all the velocities $V^{(k)}$. Then using the order of collisions, the particle locations at the $(k+1)$-th collision can be written as a linear function of the particle locations at the k-th collision. That is, $X^{(k+1)} = A^{(k)} X^{(k)} + b^{(k)}$, where $A^{(k)}$ is a matrix and $b^{(k)}$ is a vector, whose elements depend on m and e only. Because of the periodicity of the locations, we have,

$$X^{(0)} = \left[\prod_{k=0}^{K-1} A^{(k)}\right] X^{(0)} + \sum_{i=0}^{K-1}\left[\prod_{k=0}^{i-1} A^{(k)}\right] b^{(i)}.$$

After solving for $X^{(0)}$, the locations $X^{(k)}$ can be computed. Although this can be done for any sequence of collisions, it does not necessarily imply that such an orbit can be realized. This is because, the solution of the linear system may give behavior that is not physical, such as interparticle collisions that occur behind the walls. To ensure a sequence can be realized as a periodic orbit, one must ensure that all collisions occur between the two walls. One also needs to ensure that the orbit is stable. These conditions mean that a given periodic orbit can only exist in a restricted region of the (e, m) parameter space.

3. Continuous orbit transitions

We now consider continuous transitions in which we add or subtract a single collision. It is easy to see that in any sequence, there are only three types of collisions C, R and L.

When adding a C collision to an orbit, the orbit must have subsequence $CLRC$ (here the sequence is read from right to left). This is because two C collisions cannot occur consecutively. In Fig. 1, we present the sketch of the continuous transition that occurs when a C collision is added. Fig. 1(a) shows the subsequence $CLRC$, while Fig. 1(c) shows the new subsequence $CLCRC$ which arises after adding a C collision. As we will argue below, Fig. 1(b) represents the only possible continuous transition between the $CLRC$ and $CLCRC$ subsequences.

We denote the subsequence $CLRC$ by $C^{(2)}LRC^{(1)}$ to distinguish the two different interparticle collisions. In order to continuously add a C collision to the subsequence $CLRC$, the new C collision must not result in a discontinuous change in

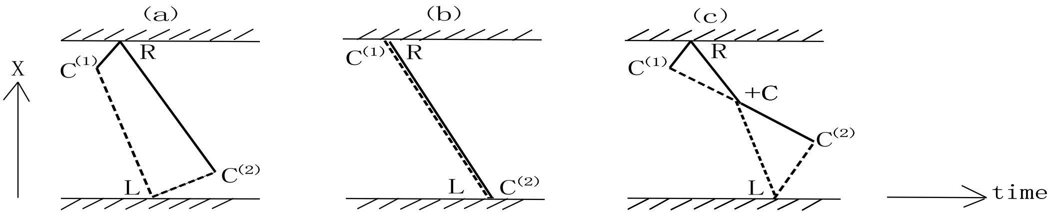

Fig. 1. The continuous transition resulting from adding C to an orbit in a two-particle system. Here we only sketch the part of the orbit where the C collision is added.

the momentum of each particle when the transition occurs. This implies that the velocities of the two particles must be equal when the continuous transition occurs. Therefore, at the point of transition, the two lines $LC^{(1)}$ and $C^{(2)}R$ in Fig. 1(b) must be parallel. It is easy to see that this can only occur if the two lines coincide. That is to say, the $C^{(1)}$ collision must occur at the right wall and $C^{(2)}$ must occur at the left wall. These two conditions represent the requirement for a continuous transition when a C collision is added. Since there are two conditions that must be satisfied, we refer to this continuous transition as being co-dimension 2. Therefore, a continuous transition must occur at a point in the 2-dimensional (e, m) parameter space. In fact, one can show that there is an infinite family of this type of transition [7].

Similarly, one can show that adding an R collision is also, generically, of co-dimension 2 [7]. However, surprisingly, there are also a number of non-generic transitions when continuously adding an R collision. These transitions will be described in detail in the next section.

For the case of adding an L collision, because of the left oscillating wall, there must be a discontinuous change in momentum of the left particle. Therefore, no continuous transition of this type can occur.

4. Degenerate transitions

From the above analysis, it seems clear that all continuous transitions should be of co-dimension 2. However, surprisingly, one can also find degenerate transitions that are of co-dimension 1.

In fact, one can show that the continuous transition between the periodic orbits $[(CR)^jCL]^N$ and $[(CR)^jCL]^{N-1}[R(CR)^jCL]$ for $j = 1, 2, \ldots, N = 1, 2, \ldots$ is a co-dimension 1 transition [7]. In the following, we give the explanation why degeneracy of this type occurs. For the orbit $(CR)^jCL$, the times that the left and the right particles need to travel between the $2i$-th and the $(2i+2)$-th collisions are $(X_1^{(2i+2)} - X_1^{(2i)})/V_1^{(2i)}$ and $(1 - X_1^{(2i)})/V_2^{(i)} + (1 - X_1^{(2i+2)})/V_2^{(2i)}$. Equating the two transition times and solving for $X_1^{(2i+2)}$, we obtain

$$X_1^{(2i+2)} = \frac{V_2^{(2i)} - V_1^{(2i)}}{V_2^{(2i)} + V_1^{(2i)}} X_1^{(2i)} + \frac{2V_1^{(2i)}}{V_2^{(2i)} + V_1^{(2i)}}.$$

If the parameters are chosen such that the system is at a transition at which an R collision is continuously added, then, by an argument similar to that given in the previous section, $X_1^{(2i)}$ must be 1. In this case, $X_1^{(2i+2)} \equiv 1$. Therefore, for the orbit $(CR)^jCL$, once the first interparticle collision occurs at the right wall, all the other interparticle collisions must occur at the right wall. One can show that in the orbit $[(CR)^jCL]^N$, all the interparticle collisions must occur at the right wall as long as one interparticle collision occurs at the right wall. Therefore, the continuous transitions between the orbits $[(CR)^jCL]^N$ and $[(CR)^jCL]^{N-1}[R(CR)^jCL]$, are degenerate and are of co-dimension 1.

In fact, one can readily show that transitions of the form $[(CR)^jCL]^N \longmapsto [(CR)^jCL]^{N-1}[R(CR)^jCL]$ are the only degenerate transitions in a two-particle system [7].

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