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Statistics

Moment inequalities for positive random variables

Inégalités de moments pour des variables aléatoires positives

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ABSTRACT

Given a random variable X, the moments, $\{m_r = EX^r\}$, satisfy

$$D_r = \det(m_{i+j}\colon\ 0 \leqslant i, j \leqslant r) \geqslant 0$$

for $r \geqslant 0$. If $X \geqslant 0$, then for $n \geqslant 1$, there is a random variable X_n such that

$$EX_n^r = m_{r+n} m_n^{-1} \Big/ \binom{n+r}{n}$$

for $r \geqslant 0$. We apply the inequality $D_r \geqslant 0$ to X_n to obtain new inequalities for the moments when $X \geqslant 0$. An application is illustrated to obtain tighter bounds for skewness and kurtosis.

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RÉSUMÉ

Soit X une variable aléatoire, les moments $\{m_r = EX^r\}$ vérifient :

$$D_r = \det(m_{i+j}\colon\ 0 \leqslant i, j \leqslant r) \geqslant 0, \quad r \geqslant 0.$$

Si $X \geqslant 0$, alors pour $n \geqslant 1$, il existe une variable aléatoire X_n telle que

$$EX_n^r = m_{r+n} m_n^{-1} \Big/ \binom{n+r}{n}, \quad r \geqslant 0.$$

On applique l'inégalité $D_r \geqslant 0$ à X_n pour obtenir de nouvelles inégalités sur les moments lorsque $X \geqslant 0$. On applique le résultat à l'obtention de bornes plus précises du coefficient de dissymétrie et du coefficient d'applatissement.

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1. Introduction

Moment inequalities have a variety of uses in reliability, to prove weak convergence of random variables, in regression and time series models, for solutions of stochastic differential equations, to estimation and model selection, to hypothesis testing applications and to kernel estimation. For most comprehensive accounts of the theory and applications of moment inequalities, see Chapter 1 of Shao [1] and Chapter 2 of Bulinski and Shashkin [2].

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Here, we provide a set of new inequalities for the raw moments of any *positive* random variable X with finite moments $\{m_r = EX^r\}$. These are given in Section 2. An application of them to obtain tighter bounds for skewness and kurtosis, two important measures of shape, is illustrated in Section 3.

2. New inequalities

Let X be a real random variable with moments $m_r = EX^r$ and $\mu_r = E(X - EX)^r$ for $r \geqslant 0$ assumed to exist. Set $\mathbf{M}_r = (m_{i+j}\colon\ 0 \leqslant i, j \leqslant r)$, $\mathbf{m} = (m_0, m_1, \ldots)$ and $D_r = D_r(\mathbf{m}) = \det \mathbf{M}_r$. Similarly, define $d_r = D_r(\boldsymbol{\mu})$ for $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots)$. Set $S = \sum_{i=0}^{r} z_i X^i$. Then $0 \leqslant E\ S^2 = \mathbf{z}'\mathbf{M}_r\mathbf{z}$ for all $\mathbf{z} = (z_0, \ldots, z_r)'$ in R^{r+1}. So,

$$D_r \geqslant 0, \tag{1}$$

and $D_r > 0$ unless X is restricted to having at most r values. We also have the result that $\det \mathbf{M}_r^* \geqslant 0$, where $\mathbf{M}_r^*$ is $\mathbf{M}_r$ with any set of rows deleted and the same set of columns deleted. The inequality (1) is well known (Bulinski and Shashkin, [2]).

Suppose now $X \geqslant 0$. For $n = 0, 1, \ldots$ let Y_n be a random variable with distribution

$$\int_0^x y^n \, \mathrm{d}P(X \leqslant y)/m_n,$$

and set $X_n = (1 - U^{1/n})Y_n$, where $U \sim U(0,1)$ independently of Y_n. So, $X_0 = Y_0$ has the same distribution as X. By Theorem 3 of Artikis and Voudouri [3],

$$E\exp(tX_n) = \left\{E\exp(tX) - \sum_{k=0}^{n-1} t^k m_k/k!\right\}/\left(t^n m_n/n!\right), \tag{2}$$

where the moment generating functions of X and X_n assumed to exist for t in the neighborhood of zero. Artikis and Voudouri [3] actually have $\sum_{k=0}^{n}$ in (2) but this is not correct. It follows from (2) that

$$EX_n^r = m_{r+n} m_n^{-1} \Big/ \binom{n+r}{n} = m_{r,n}$$

say. Substituting $\{m_{r,n}\}$ for $\{m_r\}$ into $D_r(\mathbf{m}) = D_r(m_0, m_1, m_2, \ldots)$, (1) gives a new set of inequalities,

$$D_{r,n} = D_r(m_{0,n}, m_{1,n}, m_{2,n}, \ldots) \geqslant 0,$$

for $r \geqslant 0$ and $n \geqslant 0$. Unlike $D_r = D_{r,0}$, for $n \geqslant 1$, expressing $D_{r,n}$ as a function of $(m_1, \mu_2, \mu_3, \ldots)$ gives a *longer* expression. The first few $m_{r,n}$, $D_{r,n}$ are as follows:

$$\begin{aligned}
&m_{1,1} = m_2/(2m_1), && m_{1,2} = m_3/(3m_2), && m_{1,3} = m_4/(4m_3), && \ldots,\\
&m_{2,1} = m_3/(3m_1), && m_{2,2} = m_4/(6m_2), && m_{2,3} = m_5/(10m_3), && \ldots,\\
&m_{3,1} = m_4/(4m_1), && m_{3,2} = m_5/(10m_2), && m_{3,3} = m_6/(20m_3), && \ldots,\\
&m_{4,1} = m_5/(5m_1), && m_{4,2} = m_6/(15m_2), && m_{4,3} = m_7/(35m_3), && \ldots,
\end{aligned}$$

and

$$\begin{aligned}
&D_{1,1} = m_1^{-2}\left(4m_1m_3 - 3m_2^2\right)/12,\\
&D_{1,2} = m_2^{-2}\left(3m_2m_4 - 2m_3^3\right)/18,\\
&D_{1,3} = m_3^{-2}\left(8m_3m_5 - 5m_4^2\right)/80,\\
&D_{1,4} = m_4^{-2}\left(5m_4m_6 - 3m_5^2\right)/75, \qquad \ldots,\\
&2160m_1^3 D_{2,1} = 36\left(4m_1m_3 - 3m_2^2\right)m_5 - 135m_1m_4^2 + 180m_2m_3m_4 - 80m_3^3,\\
&5400m_2^3 D_{2,2} = 20\left(3m_2m_4 - 2m_3^2\right)m_6 - 54m_2m_5^2 + 60m_3m_4m_5 - 25m_4^3,\\
&14\,000m_3^3 D_{2,3} = 5\left(8m_3m_5 - 5m_4^2\right)m_7 - 35m_3m_6^2 + 35m_4m_5m_6 - 14m_5^3,\\
&330\,750m_4^3 D_{2,4} = 63\left(5m_4m_6 - 3m_5^2\right)m_8 - 270m_4m_7^2 + 252m_5m_6m_7 - 98m_6^3,
\end{aligned}$$

and so on.

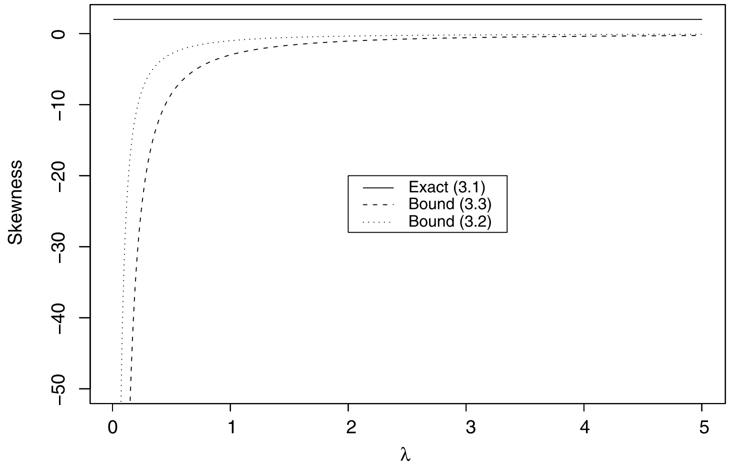

Fig. 1. The exact and approximate values of skewness for an exponential random variable with scale parameter λ.

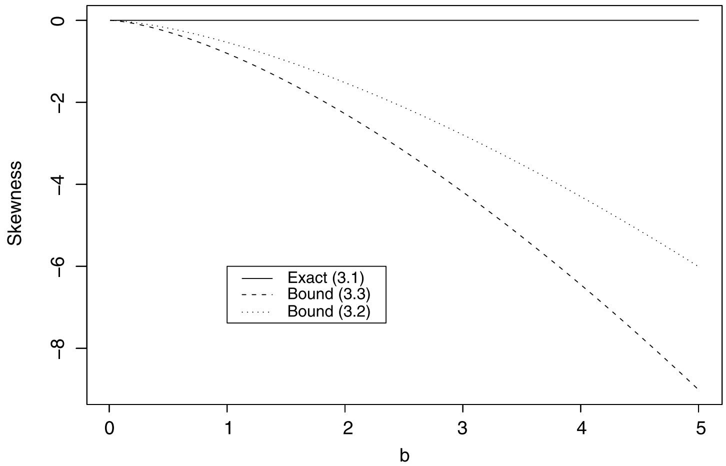

Fig. 2. The exact and approximate values of skewness for a uniform random variable on the interval $[0, b]$.

3. Application

The inequalities of Section 2 can be used to obtain tighter bounds for various statistical measures. Two important measures of the shape of a distribution are skewness and kurtosis.

The skewness measure of a random variable X is defined by,

$$\gamma_1 = \frac{m_3 - 3\mu\sigma^2 - \mu^3}{\sigma^{3/2}}, \tag{3}$$

where $\mu = m_1$ is the mean and $\sigma^2 = m_2 - m_1^2$ is the variance. Using the derived inequality, $D_{1,1} \geqslant 0$, that is $4m_1m_3 \geqslant 3m_2^2$, we obtain the bound

$$\gamma_1 \geqslant \frac{3(\sigma^2 + \mu^2)^2 - 12\mu^2\sigma^2 - 4\mu^4}{4\mu\sigma^{3/2}}. \tag{4}$$

Using the standard inequality, $m_3 \geqslant m_1^3$, we obtain the bound

$$\gamma_1 \geqslant -3\mu\sigma^{1/2}. \tag{5}$$

We claim that the bound given by (4) is tighter than (5) at least for most of the useful random variables. To verify this claim, we consider the two simplest positive random variables:

- X is an exponential random variable with scale parameter λ;
- X is a uniform random variable on the interval $[0, b]$.

The plots of the bounds given by (4) and (5) versus the exact value of skewness in (3) are shown in Figs. 1 and 2. It is clear that the bound given by (4) is tighter for both random variables. This numerical check was repeated for many other standard positive random variables. Each check showed that the bound given by (4) was tighter. The details are not given here because of space limitations.

The kurtosis measure of a random variable X is defined by,

$$\gamma_2 = \frac{m_4 - 4\mu m_3 + 6\mu^2\sigma^2 + 3\mu^4}{\sigma^2}. \tag{6}$$

Using the derived inequality, $D_{1,2} \geqslant 0$, that is $3m_2m_4 \geqslant 2m_3^3$, we obtain the bound

$$\gamma_2 \geqslant \frac{2m_3^3 - 12\mu m_2 m_3 + 18\mu^2\sigma^2 m_2 + 9\mu^4 m_2}{3\sigma^2 m_2}. \tag{7}$$

Using the standard inequality, $m_4 \geqslant m_1^4$, we obtain the bound

$$\gamma_2 \geqslant \frac{-4\mu m_3 + 6\mu^2\sigma^2 + 4\mu^4}{\sigma^2}. \tag{8}$$

We compared numerically the bounds given by (7) and (8) versus the exact value of kurtosis in (6) for X exponential, uniform and other standard positive random variables. The conclusions were the same as for the skewness measure.

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