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Partial Differential Equations

# Regularity theorems, up to the boundary, for shear thickening flows ☆

## *Théorèmes de régularité, jusqu'à la frontière des solutions de problèmes aux limites pour des fluides visqueux dilatants*

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A B S T R A C T

This Note concerns the regularity up to the boundary of weak solutions to systems describing the flow of generalized Newtonian shear thickening fluids under the homogeneous Dirichlet boundary condition. The extra stress tensor $\mathcal{S}(D)$, see (2) below, is given by a power law with shear exponent $p \geqslant 2$. Complete proofs of the results presented here are given in the forthcoming paper [4] (H. Beirão da Veiga et al., in press). The aim of this Note is to describe the results proved in H. Beirão da Veiga et al. (in press) [4], together with suitable comments.



R É S U M É

Dans cette Note on étudie la régularité jusqu'à la frontière des solutions faibles de systèmes décrivant le mouvement de fluides newtoniens généralisés visqueux dilatants, dans le cas de conditions aux limites de Dirichlet homogènes. Le tenseur des contraintes supplémentaires $\mathcal{S}(D)$, voir (2), est donné par une loi de puissance avec un exposant $p \geqslant 2$. Des résultats détaillés présentés ici sont donnés dans un article à paraître [4] (H. Beirão da Veiga et al., in press). Dans cette Note on se limite à l'énoncé des résultats démontrés dans [4] (H. Beirão da Veiga et al., in press) suivis de commentaires.



## 1. Introduction

This Note is concerned with the system

$$\begin{aligned} &\partial_t u - \operatorname{div} \mathcal{S}(Du) + [\nabla u]u + \nabla \pi = f, \\ &\operatorname{div} u = 0, \end{aligned} \tag{1}$$

describing the motion of a fluid in $\Omega \times I$, where $\Omega \subset \mathbb{R}^n$ and $I$ is interval $(0, T)$. The unknowns are the velocity $u : \Omega \times I \to \mathbb{R}^n$ and the pressure $\pi : \Omega \times I \to \mathbb{R}$. The external force $f$ is given, and the convective term is defined as

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$([\nabla u]u)_i = \sum_{j=1}^n u_j \partial_j u_i$, $i = 1, \ldots, n$. The extra stress tensor $\mathcal{S}$, expressing the inner properties of the fluid, depends only on $Du = (\nabla u + \nabla u^T)/2$ by the principle of material frame indifference, and is assumed to possess *p-structure* with $p \geqslant 2$. Under these assumptions the system (1) models the flow of shear thickening fluids. Typical examples are

$$\mathcal{S}(D) = \left(1 + |D|^2\right)^{\frac{p-2}{2}} D \quad \text{or} \quad \mathcal{S}(D) = \left(1 + |D|\right)^{p-2} D. \tag{2}$$

We assume the non-slip boundary condition on $\partial\Omega \times I$

$$u = 0. \tag{3}$$

We do not impose an initial condition since the results are local in time.

The system (1) is nowadays classical. It was proposed by O.A. Ladyzhenskaya (see [11] and references) as a modification of the Navier–Stokes system. J.-L. Lions (see [12]) suggested a similar system, however with the elliptic term depending on the full gradient. In spite of the fact that the system has been extensively studied, there are still many open problems, especially concerning the regularity of weak solutions. Our results contribute to the field of regularity properties of weak solutions to the problem (1) and its steady version (4). In particular, we prove boundary regularity of weak solutions $(u, \pi)$ of the problems (1), (2), and (3) for suitable $p \geqslant 2$. The main obstacles arise close to the boundary. They are connected with the structure of the elliptic term, namely that it is nonlinear and depends on the symmetric part of the gradient only, and with the presence of pressure in the equation. This is why we do not get the same regularity as for solutions of Stokes problem. The difficulties due to convective term are avoided by considering sufficiently large $p \geqslant 2$.

The question of regularity of weak solutions of (1) is transferred, by appealing to [6], to the problem of the regularity of weak solutions to the stationary variant of (1) in $\Omega$, namely

$$\begin{aligned} &-\operatorname{div} \mathcal{S}(Du) + \delta[\nabla u]u + \nabla\pi = f, \\ &\operatorname{div} u = 0, \end{aligned} \tag{4}$$

where $\delta = 0$ or 1. Since the problem with $\delta = 0$ already contains the principle features that appear due to the form of $\mathcal{S}$, we treat the regularity of solutions to (4) with $\delta = 0$ as the basic step of the proof.

Let us now formulate precisely the assumptions on $\mathcal{S}$.

**Assumption 1** *(extra stress tensor).* We assume that the extra stress tensor $\mathcal{S} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ belongs to $C^0(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}_{\mathrm{sym}}) \cap C^1(\mathbb{R}^{n\times n} \setminus \{0\}, \mathbb{R}^{n\times n}_{\mathrm{sym}})$, where $\mathbb{R}^{n\times n}_{\mathrm{sym}} := \{A \in \mathbb{R}^{n\times n} \mid A = A^\top\}$, and satisfies $\mathcal{S}(0) = 0$ and $\mathcal{S}(A) = \mathcal{S}(A^{\mathrm{sym}})$, where $A^{\mathrm{sym}} := \frac{1}{2}(A + A^\top)$. Moreover, we assume that $\mathcal{S}$ has $p$-structure, i.e., there exists $p \in (1, \infty)$ such that

$$\sum_{i,j,k,l=1}^n \partial_{kl}\mathcal{S}_{ij}(A) B_{ij} B_{kl} \geqslant c\left(1 + \left|A^{\mathrm{sym}}\right|\right)^{p-2} \left|B^{\mathrm{sym}}\right|^2, \tag{5a}$$

$$\left|\partial_{kl}\mathcal{S}_{ij}(A)\right| \leqslant C\left(1 + \left|A^{\mathrm{sym}}\right|\right)^{p-2} \tag{5b}$$

is satisfied for all $A, B \in \mathbb{R}^{n\times n}$ with $A^{\mathrm{sym}} \neq 0$.

These assumptions are motivated by the typical prototypes for the extra stress tensor given in (2). We refer the reader to [13,8,5] for a more detailed discussion leading to Assumption 1. We also define a function $\mathcal{F} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$, closely related to the extra stress tensor $\mathcal{S}$, by

$$\mathcal{F}(A) := \left(1 + \left|A^{\mathrm{sym}}\right|\right)^{\frac{p-2}{2}} A^{\mathrm{sym}}. \tag{6}$$

Note that $\mathcal{F}(D) \sim |D|^{p/2}$. Since in the following and in Ref. [4] we insert into $\mathcal{S}$ and $\mathcal{F}$ only symmetric tensors, we can drop the superscripts "sym" and restrict the admitted tensors to symmetric ones.

## 2. Main results

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with $C^{2,1}$ boundary, let $\mathcal{S}$ satisfy Assumption* 1 *with $p \geqslant 2$, and let $f \in L^2(\Omega)$. Then the unique weak solution $u \in W_0^{1,p}(\Omega)$, with $\operatorname{div} u = 0$, of problem* (4)*, with $\delta = 0$, and boundary condition* (3) *satisfies*

$$u \in W^{1,q}(\Omega), \qquad \mathcal{F}(Du) \in W^{1,\frac{2q}{p+q-2}}(\Omega), \tag{7}$$

*for $q = (np + 2 - p)/(n - 2)$ if $n \geqslant 3$, and for all $q < +\infty$ if $n = 2$.*

**Theorem 2.2.** *Let $\Omega$ and $f$ be as in Theorem* 2.1*, and let $\mathcal{S}$ satisfy Assumption* 1 *with $p \geqslant \max\{2, (3n)/(n + 2)\}$. Then, for $p \in [3, \infty) \cup (n/2, \infty)$, any weak solution $u \in W_0^{1,p}(\Omega)$, with $\operatorname{div} u = 0$, of problem* (4)*, with $\delta = 1$, and boundary condition* (3) *satisfies* (7)*, where $q$ is as above.*

**Remark 2.1.** (i) In the interior and in tangential directions we get better regularity properties in the previous theorems. More precisely

$$\mathcal{F}(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega), \qquad \xi\,\partial_\tau \mathcal{F}(Du) \in L^2(\Omega),$$

where $\xi$ is some cut-off function with support near the boundary $\partial\Omega$ and the tangential derivative $\partial_\tau$ is defined in a suitable neighborhood of the boundary.

(ii) One has

$$\big|\nabla \mathcal{F}(Du)\big| \simeq \big(1+|Du|\big)^{\frac{p-2}{2}} |\nabla Du|$$

and also $|\partial_\tau \mathcal{F}(Du)| \simeq (1+|Du|)^{\frac{p-2}{2}} |D\partial_\tau u|$. For $p \geqslant 2$ it follows that

$$\big|\nabla^2 u\big| \leqslant c\big|\nabla \mathcal{F}(Du)\big|.$$

**Theorem 2.3.** *Let $\mathcal{S}$ satisfy Assumption* 1 *with $p > (3n+2)/(n+2)$. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with $C^{2,1}$ boundary, $I = (0,T)$, $T > 0$, and $f \in W^{1,2}_{\mathrm{loc}}(I, L^2(\Omega))$. Then any weak solution $u \in L^\infty(I, L^2(\Omega)) \cap L^p(I, W^{1,p}_0(\Omega))$, with $\operatorname{div} u = 0$, of the problem* (1)*, and boundary condition* (3) *satisfies*

$$\begin{aligned} &u \in W^{1,\infty}_{\mathrm{loc}}\big(I, L^2(\Omega)\big), \qquad \mathcal{F}(Du) \in W^{1,2}_{\mathrm{loc}}\big(I, L^2(\Omega)\big),\\ &u \in L^\infty_{\mathrm{loc}}\big(I, W^{1,q}(\Omega)\big), \qquad \mathcal{F}(Du) \in L^\infty_{\mathrm{loc}}\big(I, W^{1,\frac{2q}{p+q-2}}(\Omega)\big) \end{aligned} \tag{8}$$

*for $q = (np+2-p)/(n-2)$ ($q < +\infty$ if $n = 2$).*

**Remark 2.2.** Note that Remark 2.1 also applies to Theorem 2.3. In particular we get

$$\mathcal{F}(Du) \in W^{1,2}_{\mathrm{loc}}(I \times \Omega), \qquad \xi\,\partial_\tau \mathcal{F}(Du) \in L^\infty_{\mathrm{loc}}\big(I, L^2(\Omega)\big).$$

## 3. Further remarks

The main obstacle in the proofs of the above theorems is the boundary condition prescribed on the non-flat boundary $\partial\Omega$ together with the fact that the extra stress tensor $\mathcal{S}$ depends only on symmetric part of the velocity gradient. Moreover, the incompressibility of the flow (which results in the fact that weak solutions are divergence free, and in the appearance of the pressure term $\nabla\pi$) causes additional problems. Let us now briefly describe how we treat these difficulties in the case of the steady problem with $\delta = 0$.

The regularity of the tangential derivatives $\partial_\tau u$ near the boundary, namely

$$\int_\Omega \xi\big|\partial_\tau \mathcal{F}(Du)\big|^2\,\mathrm{d}x \simeq \int_\Omega \xi\big(1+|Du|\big)^{p-2} |D\partial_\tau u|^2\,\mathrm{d}x < C, \tag{9}$$

is obtained by the classical difference quotients method. We appeal to translations parallel to the non-flat boundary $\partial\Omega$. When deriving the main estimates there appear some terms which are not present if the boundary is flat. The normal derivatives are restored from Eqs. (1). Due to the smoothness of $\partial\Omega$, it is possible to express the whole second gradient of $u$ via the gradient of the tangential derivatives of $u$ and $\pi$. Here, in appealing to (9), one has to take into account that we do not know if, for $p \neq 2$, a Korn's inequality, of the type

$$\int_\Omega \big(1+|Du|\big)^{p-2} |\nabla\partial_\tau u|^2\,\mathrm{d}x \leqslant C\int_\Omega \big(1+|Du|\big)^{p-2} |D\partial_\tau u|^2\,\mathrm{d}x,$$

holds. This situation leads to a lower regularity in the normal direction.

The results in this article are new and improve all the previous results. They are obtained by combining several methods from [10,14,1,3] with some new ideas. As in [10], [14], and [2], we consider non-flat boundaries. In particular, we improve the regularity exponents obtained in [2] by finding a better balance between the two main terms which prevent optimal results.

The treatment of the unsteady problem (1) is based on the results for the steady problem (4) and on the improvements of the time regularity of weak solutions shown in [6].

Apart from the results mentioned above, there are some other investigations dealing with the boundary regularity of weak solutions to systems related to (1). For some references we refer the reader to [4]. We just mention here the forthcoming paper [7], where the authors prove, for small data $f$, that the solutions belong to $C^{1,\alpha}(\overline{\Omega}) \cap W^{2,2}(\Omega)$, up to the boundary. Finally we claim that the method followed in [4] may lead to an improvement of the result in [9] for all $p \geqslant 2$.

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