



Probability Theory

Uniqueness of solutions for multidimensional BSDEs with uniformly continuous generators [☆]

Unicité des solutions d'équations différentielles stochastiques multidimensionnelle ayant des générateurs uniformément continus

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ABSTRACT

Hamadène (2003) [2] obtained an existence result of solutions for multidimensional backward stochastic differential equations (BSDEs) with uniformly continuous generators, provided that the i th component $g_i(t, y, z)$ of the generator g depends only on the i th row of the matrix z . The uniqueness of solutions for this kind of BSDEs is proved in this Note.

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R É S U M É

Hamadène (2003) [2] a obtenu un résultat d'existence des solutions d'équations différentielles stochastiques rétrogrades (BSDE) à générateurs uniformément continus à condition que la i -ème composante $g_i(t, y, z)$ du générateur g ne dépende que de la i -ème ligne de la matrice z . L'unicité de la solution pour ce type d'équations BSDE est démontrée dans cette Note.

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1. Introduction

In this Note, we consider the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad t \in [0, T], \quad (1)$$

where $T > 0$ is a constant termed the time horizon, ξ is a k -dimensional random vector termed the terminal condition, the random function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k \times d} \rightarrow \mathbf{R}^k$ is progressively measurable for each (y, z) , termed the generator of the BSDE (1), and B is a d -dimensional Brownian motion. The solution (y, z) is a pair of adapted processes. The triple (ξ, T, g) is called the coefficients (parameters) of the BSDE (1).

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Such equations, in the nonlinear case, were firstly introduced by Pardoux and Peng [5], who established an existence and uniqueness result for solutions to BSDEs under the Lipschitz assumption of the generator g . Since then, BSDEs have been studied with great interest, and they have gradually become an important mathematical tool in many fields such as financial mathematics, stochastic games and optimal control, etc. In particular, many efforts have been done in relaxing the Lipschitz hypothesis on g , for instance, Mao [3] proved an existence and uniqueness result of a solution for (1) where g satisfies some kind of non-Lipschitz conditions, Pardoux [4] established an existence and uniqueness result of a solution for (1) where g satisfies some kind of monotonicity conditions in y , and Hamadène [2] obtained an existence result of a solution for (1) where g is uniformly continuous in (y, z) and for each $i = 1, 2, \dots, k$, the i th component $g_i(t, y, z)$ of g depends only on the i th row of the matrix z .

This Note aims at proving the uniqueness of a solution for BSDEs under the same assumptions as those in Hamadène [2].

2. Main result and its proof

Let us first introduce some notations. First of all, let us fix a number $T > 0$, and two positive integers k and d . Let (Ω, \mathcal{F}, P) be a probability space carrying a standard d -dimensional Brownian motion $(B_t)_{t \geq 0}$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural σ -algebra generated by $(B_t)_{t \geq 0}$ and $\mathcal{F} = \mathcal{F}_T$. In this paper, the Euclidean norm of a vector $y \in \mathbf{R}^k$ will be defined by $|y|$, and for an $k \times d$ matrix z , we define $|z| = \sqrt{\text{Tr}(zz^*)}$, where z^* is the transpose of z . Let $\langle x, y \rangle$ represent the inner product of $x, y \in \mathbf{R}^k$. We denote by $L^2(\mathcal{F}_T; \mathbf{R}^k)$ the set of all \mathbf{R}^k -valued, square integral and \mathcal{F}_T -measurable random vectors. Let $\mathcal{S}^2(0, T; \mathbf{R}^k)$ denote the set of \mathbf{R}^k -valued, adapted and continuous processes $(\phi_t)_{t \in [0, T]}$ such that $\|\phi\|_{\mathcal{S}^2}^2 := \mathbf{E}[\sup_{t \in [0, T]} |\phi_t|^2] < +\infty$. Moreover, let $M^2(0, T; \mathbf{R}^{k \times d})$ denote the set of (equivalent classes of) \mathcal{F}_t -progressively measurable $\mathbf{R}^{k \times d}$ -valued processes $(\varphi_t)_{t \in [0, T]}$ such that $\|\varphi\|_{M^2}^2 := \mathbf{E}[\int_0^T |\varphi_t|^2 dt] < +\infty$. Obviously, $\mathcal{S}^2(0, T; \mathbf{R}^k)$ is a Banach space and $M^2(0, T; \mathbf{R}^{k \times d})$ is a Hilbert space.

As mentioned in the introduction, we will deal only with BSDEs which are equations of type (1), where the terminal condition $\xi \in L^2(\mathcal{F}_T; \mathbf{R}^k)$, and the generator g is (\mathcal{F}_t) -progressively measurable for each (y, z) .

Definition 1. A pair of processes $(y_t, z_t)_{t \in [0, T]}$ is called a solution to the BSDE (1), if $(y_t, z_t)_{t \in [0, T]} \in \mathcal{S}^2(0, T; \mathbf{R}^k) \times M^2(0, T; \mathbf{R}^{k \times d})$ and satisfies the BSDE (1).

Let us firstly introduce the following assumptions:

(H1) g is uniformly continuous in y uniformly with respect to (ω, t, z) , i.e., there exists a continuous nondecreasing function $\rho(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with at most linear growth and satisfying $\rho(0) = 0$ and $\rho(u) > 0$ for $u > 0$ such that $dP \times dt - a.s.$,

$$\forall y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \rho(|y_1 - y_2|).$$

Moreover, we assume that $\int_{0+} \frac{du}{\rho(u)} = +\infty$.

(H2) g is uniformly continuous in z uniformly with respect to (ω, t, y) , i.e., there exists a continuous nondecreasing function $\phi(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^+$ with at most linear growth and satisfying $\phi(0) = 0$, such that $dP \times dt - a.s.$,

$$\forall y \in \mathbf{R}^k, z_1, z_2 \in \mathbf{R}^{k \times d}, \quad |g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \phi(|z_1 - z_2|).$$

(H3) For any $i = 1, \dots, k$, $g_i(t, y, z)$, the i th component of g , depends only on the i th row of the matrix z .

$$(H4) \quad \mathbf{E} \left[\left(\int_0^T |g(t, 0, 0)| dt \right)^2 \right] < +\infty.$$

In the sequel, we denote the constant of the linear growth for $\rho(\cdot)$ and $\phi(\cdot)$ in (H1) and (H2) by $A > 0$, i.e., $\rho(x) \leq A(x+1)$ and $\phi(x) \leq A(x+1)$ for $x \geq 0$.

The following Theorem 1 is the main result of this Note:

Theorem 1. Let g satisfy the assumptions (H1)–(H4). Then for each $\xi \in L^2(\mathcal{F}_T; \mathbf{R}^k)$, the BSDE with parameters (ξ, T, g) has a unique solution.

Proof of Theorem 1. Hamadène [2] has given the proof for the existence part of Theorem 1. Now, let us prove the uniqueness part of Theorem 1. Let both $(y_t, z_t)_{t \in [0, T]}$ and $(y'_t, z'_t)_{t \in [0, T]}$ be the solution of the BSDE with parameters (ξ, T, g) .

(i) We prove that the process $y_t - y'_t$ is uniformly bounded, i.e., there exists a constant $C > 0$ such that $dP \times dt - a.s.$, $|y_t - y'_t| \leq C$. Indeed, using Itô's formula to $|y_t - y'_t|^2$ we arrive, for each $t \in [0, T]$, at

$$|y_t - y'_t|^2 + \int_t^T |z_s - z'_s|^2 ds = 2 \int_t^T (y_s - y'_s, g(s, y_s, z_s) - g(s, y'_s, z'_s)) ds - 2 \int_t^T (y_s - y'_s, (z_s - z'_s) dB_s).$$

It follows from (H1) and (H2) that

$$\begin{aligned} 2|y_s - y'_s, g(s, y_s z_s) - g'(s, y'_s, z'_s)| &\leq 2|y_s - y'_s| \{ \rho(|y_s - y'_s|) + \phi(|z_s - z'_s|) \} \\ &\leq 2A|y_s - y'_s| \{ |y_s - y'_s| + |z_s - z'_s| + 2 \} \\ &\leq (2A + 2A^2 + 1)|y_s - y'_s|^2 + |z_s - z'_s|^2/2 + 4A^2. \end{aligned}$$

On the other hand, since both (y_t, z_t) and (y'_t, z'_t) belong to the process space $\mathcal{S}^2(0, T; \mathbf{R}^k) \times \mathcal{M}^2(0, T; \mathbf{R}^{k \times d})$, then, using BGD's inequality we deduce that $(\int_0^t (y_s - y'_s, (z_s - z'_s) dB_s))_{t \in [0, T]}$ is an (\mathcal{F}_t, P) -martingale. Thus, for each $0 \leq t \leq s \leq T$, we have

$$\mathbf{E}[|y_s - y'_s|^2 | \mathcal{F}_t] \leq (2A + 2A^2 + 1) \int_s^T \mathbf{E}[|y_u - y'_u|^2 | \mathcal{F}_t] du + 4A^2 T.$$

By Gronwall's inequality we have $\mathbf{E}[|y_s - y'_s|^2 | \mathcal{F}_t] \leq 4A^2 T e^{(2A+2A^2+1)T}$, which yields the desired result after taking $s = t$.

(ii) We show that the solution of the BSDE with parameters (ξ, g, T) is unique. For each $i = 1, \dots, k$ and $t \in [0, T]$, by (H3) we have

$${}^i y_t - {}^i y'_t = \int_t^T (g_i(s, y_s, {}^i z_s) - g_i(s, y'_s, {}^i z'_s)) ds - \int_t^T ({}^i z_s - {}^i z'_s) dB_s$$

where ${}^i y_t, {}^i y'_t, g_i, {}^i z_t$ and ${}^i z'_t$ are the i th components and rows of respectively y_t, y'_t, g, z_t and z'_t . Then, using Tanaka's formula we obtain that for each $t \in [0, T]$,

$$|{}^i y_t - {}^i y'_t| \leq \int_t^T \text{sgn}({}^i y_s - {}^i y'_s) (g_i(s, y_s, {}^i z_s) - g_i(s, y'_s, {}^i z'_s)) ds - \int_t^T \text{sgn}({}^i y_s - {}^i y'_s) ({}^i z_s - {}^i z'_s) dB_s. \tag{2}$$

It follows from (H1) and (H2) that

$$|g_i(s, y_s, {}^i z_s) - g_i(s, y'_s, {}^i z'_s)| \leq \rho(|y_s - y'_s|) + \phi(|{}^i z_s - {}^i z'_s|). \tag{3}$$

Furthermore, recalling that $\phi(\cdot)$ is a non-decreasing function from \mathbf{R}^+ to itself with at most linear growth, one knows from Fan and Jiang [1] that for each $m \geq 1$,

$$\phi(x) \leq (m + 2A)x + \phi\left(\frac{2A}{m + 2A}\right) \tag{4}$$

holds true for each $x \in \mathbf{R}^+$. Thus, combining (2), (3) and (4) we know that for each $m \geq 1$,

$$|{}^i y_t - {}^i y'_t| \leq \phi\left(\frac{2A}{m + 2A}\right) T + \int_t^T (\rho(|y_s - y'_s|) + (m + 2A)|{}^i z_s - {}^i z'_s|) ds - \int_t^T \text{sgn}({}^i y_s - {}^i y'_s) ({}^i z_s - {}^i z'_s) dB_s. \tag{5}$$

Now, for each $t \in [0, T]$, let $a_t^{m,i} = (m + 2A) \frac{\text{sgn}({}^i y_t - {}^i y'_t) ({}^i z_t - {}^i z'_t)^*}{|{}^i z_t - {}^i z'_t|} 1_{|{}^i z_t - {}^i z'_t| \neq 0}$, then $(a_t^{m,i})_{t \in [0, T]}$ is an \mathbf{R}^d -valued, bounded and (\mathcal{F}_t) -adapted process. By Girsanov's theorem, we know that $\bar{B}_t^{m,i} = B_t - \int_0^t a_s^{m,i} ds, t \in [0, T]$, is a d -dimensional Brownian motion under the probability $\bar{P}^{m,i}$ on (Ω, \mathcal{F}) defined by: $\frac{d\bar{P}^{m,i}}{dP} = \exp[\int_0^T (a_s^{m,i})^* dB_s - \frac{1}{2} \int_0^T |a_s^{m,i}|^2 ds]$. Moreover, the process $(\int_0^t \text{sgn}({}^i y_s - {}^i y'_s) ({}^i z_s - {}^i z'_s) d\bar{B}^{m,i}(s))_{0 \leq t \leq T}$ is an $(\mathcal{F}_t, \bar{P}^{m,i})$ -martingale. Indeed, let $\bar{\mathbf{E}}^{m,i}[X]$ represent the expectation of the random variable X under $\bar{P}^{m,i}$, then from BGD's inequality and Hölder's inequality we have:

$$\begin{aligned} &\bar{\mathbf{E}}^{m,i} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \text{sgn}({}^i y_s - {}^i y'_s) ({}^i z_s - {}^i z'_s) d\bar{B}^{m,i}(s) \right| \right] \\ &\leq K \bar{\mathbf{E}}^{m,i} \left[\sqrt{\int_0^T |{}^i z_s - {}^i z'_s|^2 ds} \right] \leq K \sqrt{\mathbf{E} \left[\left(\frac{d\bar{P}^{m,i}}{dP} \right)^2 \right]} \sqrt{\mathbf{E} \left[\int_0^T |{}^i z_s - {}^i z'_s|^2 ds \right]} < +\infty, \end{aligned}$$

where K is a constant. Thus, from (5) we obtain that for each $m \geq 1$,

$$|y_t - y'_t| \leq \phi\left(\frac{2A}{m+2A}\right)T + \int_t^T \rho(|y_s - y'_s|) ds - \int_t^T \operatorname{sgn}(y_s - y'_s)(z_s - z'_s) d\bar{B}^{m,i}(s),$$

and then, for each $m \geq 1$, $i = 1, \dots, k$ and each $0 \leq t \leq s \leq T$,

$$\bar{\mathbb{E}}^{m,i}[|y_s - y'_s| | \mathcal{F}_t] \leq \phi\left(\frac{2A}{m+2A}\right)T + \int_t^T \bar{\mathbb{E}}^{m,i}[\rho(|y_u - y'_u|) | \mathcal{F}_t] du. \quad (6)$$

In the following, for each $n \geq 1$, let $\rho_n : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $\rho_n(x) = \sup_{y \in \mathbf{R}^+} \{\rho(y) - n|x - y|\}$. Since the rate of growth of ρ is at most linear, ρ_n is defined and Lipschitz. Moreover, the sequence $(\rho_n)_{n=1}^{+\infty}$ is non-increasing and converges to ρ . For each $m \geq 1$ and $n \geq 1$, let $v_m^n : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be the solution of the following backward ordinary differential equation:

$$v_m^n(t) = \phi\left(\frac{2A}{m+2A}\right)T + \int_t^T \rho_n(k \cdot v_m^n(s)) ds, \quad t \in [0, T].$$

Since $(\rho_n)_n$ is a non-increasing sequence, $v_m^{n+1} \leq v_m^n$ for any $n \geq 1$. This implies that the sequence $(v_m^n)_n$ converges pointwise to a function $v_m : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ which satisfies: $v_m(t) = \phi\left(\frac{2A}{m+2A}\right)T + \int_t^T \rho(k \cdot v_m(s)) ds$, $t \in [0, T]$. Furthermore, since $\phi(\cdot)$ is a non-increasing function, $v_{m+1} \leq v_m$ for any $m \geq 1$. This implies that, noticing that $\phi(\cdot)$ is a continuous function with $\phi(0) = 0$, the sequence $(v_m)_m$ converges pointwise to a function $v : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ which satisfies $v(t) = \int_t^T \rho(k \cdot v(s)) ds$, $t \in [0, T]$. Recalling that $\int_{0^+} \frac{du}{\rho(u)} = +\infty$, Bahari's inequality (see Lemma 3.6 in Mao [3]) yields that $v(t) = 0$ for each $t \in [0, T]$.

Now for $m, n, j \geq 1$, let $v_m^{n,j}$ be the function defined recursively as follows:

$$v_m^{n,1} = C; \quad v_m^{n,j+1}(t) = \phi\left(\frac{2A}{m+2A}\right)T + \int_t^T \rho_n(k \cdot v_m^{n,j}(s)) ds, \quad j \geq 1, t \in [0, T], \quad (7)$$

where C is defined in (i). Since ρ_n is Lipschitz, $v_m^{n,j} \rightarrow v_m^n$ as $j \rightarrow \infty$. On the other hand, it is easily seen by induction that for all $n, m, j \geq 1$,

$$|y_t - y'_t| \leq v_m^{n,j}(t), \quad t \in [0, T], i = 1, \dots, k. \quad (8)$$

Indeed, for $j = 1$ the formula holds true by (i). Suppose it also holds for each j , then for each $t \in [0, T]$, $\rho(|y_t - y'_t|) \leq \rho(k \cdot v_m^{n,j}(t)) \leq \rho_n(k \cdot v_m^{n,j}(t))$. Now, using (6) with $s = t$ and (7), we have, $\forall n, m \geq 1$, $|y_t - y'_t| \leq v_m^{n,j+1}(t)$, $t \in [0, T]$, $i = 1, \dots, k$. Thus, taking the limit in (8) as first $j \rightarrow \infty$, then $n \rightarrow \infty$, and finally $m \rightarrow \infty$, we obtain $|y_t - y'_t| = 0$ for each $t \in [0, T]$ and each $i = 1, \dots, k$. Therefore the solution is unique. The proof of Theorem 1 is then completed. \square

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