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Justification of a simplified model for shells in nonlinear elasticity

Justification d'un modèle simplifié pour les coques en élasticité non linéaire

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ABSTRACT

We introduce a simplified model for the minimization of the elastic energy in thin shells. The thickness of the shell remains a parameter in this new model.

RÉSUMÉ

Nous développons un modèle simplifié pour le problème de minimisation de l'énergie élastique d'une coque mince. L'épaisseur de la coque reste un paramètre du modèle.

Version française abrégée

Nous considérons une coque élastique d'épaisseur 2δ, de surface moyenne S, soumise à des forces extérieures $f_{\kappa,\delta}$. Pour une déformation $v : S \times]-\delta,\delta[\to \mathbf{R}^3$, l'énergie totale est donnée par $J_{\kappa,\delta}(v) = \int_{S\times]-\delta,\delta[} \widehat{W}(E(v)) - \int_{S\times]-\delta,\delta[} f_{\kappa,\delta}\cdot(v - I_d)$ si $\det(\nabla v) > 0$ où $E(v) = 1/2((\nabla v)^T\nabla v - \mathbf{I}_3)$ est le tenseur de Green–St-Venant. Les inégalités de Korn établies dans [1] (voir aussi [4]) permettent de conclure que si $f_{\kappa,\delta}$ est d'ordre δ^κ ($\kappa \geqslant 2$), alors $J_{\kappa,\delta}(v)$ est au moins d'ordre $\delta^{2\kappa-1}$. On pose

$$m_\kappa = \liminf_{\delta\to 0} \frac{m_{\kappa,\delta}}{\delta^{2\kappa-1}}, \qquad m_{\kappa,\delta} = \inf_{v\in\mathbf{U}_\delta} J_{\kappa,\delta}(v),$$

où $\mathbf{U}_\delta$ est l'ensemble des déformations admissibles. Pour une matériau de St-Venant–Kirchhoff (par exemple), l'existence de minima est un problème ouvert. Nous remplaçons alors le problème de minimisation pour $J_{\kappa,\delta}$ par un problème de minimisation pour une fonctionnelle plus simple $J^s_{\kappa,\delta}$ qui admet des minima sur un nouvel ensemble $\mathbf{D}$. L'expression de $J^s_{\kappa,\delta}$ et le choix de $\mathbf{D}$ s'appuient sur la décomposition des déformations d'une coque introduite dans [1]. Rappelons qu'une déformation v de la coque s'écrit $v = \mathcal{V} + s_3\mathbf{Rn} + \bar{v}$, où $\mathcal{V}$ et $\mathbf{R}$ sont définis sur S, s_3 est la variable suivant $\mathbf{n}$ (vecteur unitaire normal à S). Le champ $\mathbf{R}$ est à valeurs dans $SO(3)$.

L'ensemble $\mathbf{D}$ est constitué des triplets $\mathbf{v} = (\mathcal{V},\mathbf{R},\bar{v})$ admissibles (voir la Section 3). Grâce aux estimations de [1], l'expression du tenseur de Green–St-Venant $E(v)$ est simplifiée pour donner une matrice $\widehat{E}_\delta(\mathbf{v})$ dans laquelle n'apparaissent que $\frac{\partial\bar{v}}{\partial s_3}$ et les dérivées partielles premières de $\mathcal{V}$, $\mathbf{R}$ et ceci de façon linéaire (voir Section 3). L'énergie $J^s_{\kappa,\delta}(\mathbf{v})$ s'obtient en remplaçant $\int_{S\times]-\delta,\delta[} \widehat{W}(E(v))$ par $\int_{S\times]-\delta,\delta[} Q'(\widehat{E}_\delta(\mathbf{v}))$ où Q' est une forme quadratique proche de $\widehat{W}$ en 0 et en ajoutant deux termes de pénalisation afin d'approcher la condition cinématique limite $\frac{\partial\mathcal{V}}{\partial s_\alpha} = \mathbf{Rt}_\alpha$ et d'assurer la coercivité de $J^s_{\kappa,\delta}$. Nous montrons que $J^s_{\kappa,\delta}$ admet un minimum sur $\mathbf{D}$ qui est d'ordre $\delta^{2\kappa-1}$. On pose

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$$m^s_\kappa = \liminf_{\delta\to 0} \frac{m^s_{\kappa,\delta}}{\delta^{2\kappa-1}}, \qquad m^s_{\kappa,\delta} = \min_{\mathbf{v}\in\mathbf{D}} J^s_{\kappa,\delta}(\mathbf{v}).$$

Une justification de la simplification décrite ci-dessus consiste à montrer que $m^s_\kappa = \lim_{\delta\to 0} \frac{m^s_{\kappa,\delta}}{\delta^{2\kappa-1}} = \lim_{\delta\to 0} \frac{m_{\kappa,\delta}}{\delta^{2\kappa-1}} = m_\kappa$. Nous présentons ici ce résultat pour $\kappa = 2$.

1. Introduction

We consider an elastic shell of thickness 2δ submitted to applied forces. The Korn's inequalities established in [1] allow to estimate the order of the minimum of the total energy in term of the order of the forces and δ. As an example, for a St-Venant–Kirchhoff's material, proving that this minimum is achieved remains an open question. It is the object of this Note to replace the minimization problem for the total energy by a simpler one for a simplified energy which admits minimizers and to justify this approach. One of the main argument in this process is the decomposition of the deformations in the shell and the estimates derived in [1]. This allows to highlight the terms which should be neglected in the expression of the Green–St-Venant's strain tensor leading to a simplified strain tensor. In a main critical case, we prove that the asymptotic behaviors of the minima of the total energy and of the simplified one are the same.

The justification of simplified models for rods and plates in linear elasticity, based on a decomposition technique of the displacement, is presented in [6,7]. In this linear case, error estimates between the solution of the initial model and the one of the simplified model are also established (deriving such errors estimates in nonlinear elasticity is still an open problem). In some sense, these works give a mathematical justification of Timoshenko's model for straight rods and Reissner–Mindlin's model for plates (see references in [3]). The detailed proofs of the results announced in the present Note will be presented in forthcoming papers.

2. The geometry and notations

The mid-surface S of the shell is defined as $\phi(\overline{\omega})$ where ϕ is a $\mathcal{C}^2$-injective mapping from $\overline{\omega}$ into $\mathbf{R}^3$, ω being a bounded domain in $\mathbf{R}^2$ with a Lipschitzian boundary. We assume that the tangential vectors $\mathbf{t}_1 = \frac{\partial\phi}{\partial s_1}$ and $\mathbf{t}_2 = \frac{\partial\phi}{\partial s_2}$ are linearly independent at each point of $\overline{\omega}$. We set $\Omega_\delta = \omega\times\,]-\delta,\delta[$. For δ small enough, the shell $\mathcal{Q}_\delta$ is defined as $\mathcal{Q}_\delta = \Phi(\Omega_\delta)$ where Φ is the map $(s_1,s_2,s_3)\longmapsto x = \phi(s_1,s_2) + s_3\mathbf{n}(s_1,s_2)$ with $\mathbf{n} = \frac{\mathbf{t}_1\wedge\mathbf{t}_2}{\|\mathbf{t}_1\wedge\mathbf{t}_2\|_2}$. We denote by $(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})$ the 3×3 matrix with first column $\mathbf{t}_1$, second column $\mathbf{t}_2$ and third column $\mathbf{n}$.

The domain Ω_δ is rescaled into $\Omega = \omega\times\,]-1,1[$ using the operator

$$(\Pi_\delta w)(s_1,s_2,S_3) = w(s_1,s_2,\delta S_3) \quad \text{for any } (s_1,s_2,S_3)\in\Omega$$

defined e.g. for $w\in L^2(\Omega_\delta)$ for which $(\Pi_\delta w)\in L^2(\Omega)$.

By convention a field $w(x)$ for $x\in\mathcal{Q}_\delta$ is still denoted by $w(s)$ for $s\in\Omega_\delta$ $(x=\Phi(s))$. From now on, all the constants do not depend on δ.

3. Decomposition of a deformation

Let v be a deformation in $(H^1(\mathcal{Q}_\delta))^3$ such that $\|dist(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leqslant C(S)\delta^{3/2}$. In [1], it is shown that v can be decomposed as $v(s) = \mathcal{V}(s_1,s_2) + s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2) + \overline{v}(s)$ where $\mathcal{V}\in(H^1(\omega))^3$, $\mathbf{R}\in(H^1(\omega))^{3\times 3}$ such that $\mathbf{R}(s_1,s_2)\in SO(3)$ for almost all $(s_1,s_2)\in\omega$ and $\overline{v}\in(H^1(\Omega_\delta))^3$. Moreover the following estimates hold

$$\begin{aligned}
&\frac{1}{\delta}\|\overline{v}\|_{(L^2(\Omega_\delta))^3} + \|\nabla_s\overline{v}\|_{(L^2(\Omega_\delta))^9} \leqslant C\big\|dist\big(\nabla_x v, SO(3)\big)\big\|_{L^2(\mathcal{Q}_\delta)},\\
&\delta\left\|\frac{\partial\mathbf{R}}{\partial s_\alpha}\right\|_{(L^2(\omega))^9} + \left\|\frac{\partial\mathcal{V}}{\partial s_\alpha} - \mathbf{R}\mathbf{t}_\alpha\right\|_{(L^2(\omega))^3} \leqslant \frac{C}{\delta^{1/2}}\big\|dist\big(\nabla_x v, SO(3)\big)\big\|_{L^2(\mathcal{Q}_\delta)},\\
&\|\nabla_x v - \mathbf{R}\|_{(L^2(\Omega_\delta))^9} \leqslant C\big\|dist\big(\nabla_x v, SO(3)\big)\big\|_{L^2(\mathcal{Q}_\delta)}.
\end{aligned} \tag{1}$$

In order to simplify the Green–St-Venant's strain tensor, we first write the identity $(\nabla_x v)^T\nabla_x v - \mathbf{I}_3 = (\nabla_x v - \mathbf{R})^T\mathbf{R} + \mathbf{R}^T(\nabla_x v - \mathbf{R}) + (\nabla_x v - \mathbf{R})^T(\nabla_x v - \mathbf{R})$. Since indeed $\mathbf{R}$ is of order 1, estimates (1) suggest to neglect the last term in the right-hand side of the above equality whose L^1-norm is smaller than $C\delta^3$ while the two first terms are smaller than $C\delta^2$ in L^1-norm.

We have $\Pi_\delta(\nabla_x v - \mathbf{R})(\mathbf{t}_\alpha + \delta S_3\frac{\partial\mathbf{n}}{\partial s_\alpha}) = (\frac{\partial\mathcal{V}}{\partial s_\alpha} - \mathbf{R}\mathbf{t}_\alpha) + \delta S_3\frac{\partial\mathbf{R}}{\partial s_\alpha}\mathbf{n} + \Pi_\delta(\frac{\partial\overline{v}}{\partial s_\alpha})$. Then, using the L^2-estimate of $\Pi_\delta(\overline{v})$ and comparing the H^{-1}-norm of the terms in the previous equality prompts us to neglect $\Pi_\delta(\frac{\partial\overline{v}}{\partial s_\alpha})$.

Now, if in the Green–St-Venant's strain tensor of v, we carry out the simplifications mentioned above, we are brought to replace $\frac{1}{2}\Pi_\delta((\nabla_x v)^T\nabla_x v - \mathbf{I}_3)$ by $(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-T}E_\delta(v)(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-1}$ where the symmetric matrix $E_\delta(v)\in(L^2(\Omega))^{3\times 3}$ is equal to

$$\begin{pmatrix} \delta S_3 \Gamma_{11}(\mathbf{R}) + \mathcal{Z}_{11} & \delta S_3 \Gamma_{12}(\mathbf{R}) + \mathcal{Z}_{12} & \frac{1}{2\delta}\frac{\partial \overline{V}}{\partial S_3}\cdot \mathbf{t}_1 + \frac{1}{2}\mathcal{Z}_{31} \\ * & \delta S_3 \Gamma_{22}(\mathbf{R}) + \mathcal{Z}_{22} & \frac{1}{2\delta}\frac{\partial \overline{V}}{\partial S_3}\cdot \mathbf{t}_2 + \frac{1}{2}\mathcal{Z}_{32} \\ * & * & \frac{1}{\delta}\frac{\partial \overline{V}}{\partial S_3}\cdot \mathbf{n} \end{pmatrix} \tag{2}$$

and where $\Gamma_{\alpha\beta}(\mathbf{R}) = \frac{1}{2}[\frac{\partial \mathbf{R}}{\partial s_\alpha}\mathbf{n}\cdot\mathbf{Rt}_\beta + \frac{\partial \mathbf{R}}{\partial s_\beta}\mathbf{n}\cdot\mathbf{Rt}_\alpha]$, $\mathcal{Z}_{\alpha\beta} = \frac{1}{2}[(\frac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{Rt}_\alpha)\cdot\mathbf{Rt}_\beta + (\frac{\partial \mathcal{V}}{\partial s_\beta} - \mathbf{Rt}_\beta)\cdot\mathbf{Rt}_\alpha]$, $\mathcal{Z}_{3\alpha} = \frac{\partial \mathcal{V}}{\partial s_\alpha}\cdot\mathbf{Rn}$ and $\overline{V} = \Pi_\delta(\mathbf{R}^T\bar{v})$. Let us point out $E_\delta(v)$ has a linear dependence with respect to $\frac{\partial \overline{V}}{\partial S_3}$ and to the first partial derivatives of $\mathcal{V}$ and $\mathbf{R}$. Now our goal is to introduce a simplified strain tensor for a triple $(\mathcal{V}, \mathbf{R}, \overline{V})$ using the expression (2). To this end, let us introduce the set $\mathbf{D}$

$$\mathbf{D} = \Big\{ \mathbf{v} = (\mathcal{V}, \mathbf{R}, \overline{V}) \in \big(H^1(\omega)\big)^3 \times \big(H^1(\omega)\big)^{3\times 3} \times \big(L^2(\omega; H^1(-1,1))\big)^3 \;\Big|\; \mathbf{R}(s_1,s_2) \in SO(3),$$
$$\int_{-1}^{1} \overline{V}(s_1,s_2,S_3)\,\mathrm{d}S_3 = 0,\ \int_{-1}^{1} S_3 \overline{V}(s_1,s_2,S_3)\mathbf{t}_\alpha\,\mathrm{d}S_3 = 0,\ \text{for a.e. } (s_1,s_2)\in\omega,\ \alpha = 1,2 \Big\}. \tag{3}$$

For any $\mathbf{v} \in \mathbf{D}$, we define the deformation $v(s) = \mathcal{V}(s_1,s_2) + \mathbf{R}(s_1,s_2)[s_3\mathbf{n}(s_1,s_2) + \overline{V}(s_1,s_2,\frac{s_3}{\delta})] = \mathcal{V}(s_1,s_2) + s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2) + \bar{v}(s)$ for a.e. $s \in \Omega_\delta$ and we set $\widehat{E}_\delta(\mathbf{v}) = E_\delta(v)$. Let us point out that the limit kinematic condition $\frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{Rt}_\alpha$ (see e.g. [1]) is not imposed in $\mathbf{D}$ which must not be considered as the "limit set" (as $\delta \to 0$) for the triples $\mathbf{v}$. If this kinematic condition is included in the definition of $\mathbf{D}$, then the tensor $\frac{1}{\delta}\widehat{E}_\delta(\mathbf{v})$ is identical to the limit of the Green–St-Venant's strain tensor as $\delta \to 0$ (at least for $\kappa = 2$) and the quantity $\|\widehat{E}_\delta(\mathbf{v})\|^2_{(L^2(\Omega))^{3\times 3}}$ allows to control the product norm of $\mathbf{v}$ in $\mathbf{D}$. In order to get an energy which controls the product norm of $\mathbf{v}$, for any $\mathbf{v}$ in $\mathbf{D}$, we are led to add two penalization terms, which vanish as $\delta \to 0$, to $\|\widehat{E}_\delta(\mathbf{v})\|^2_{(L^2(\Omega))^{3\times 3}}$.

We set

$$\mathcal{E}_\delta(\mathbf{v}) = \big\|\widehat{E}_\delta(\mathbf{v})\big\|^2_{(L^2(\Omega))^{3\times 3}} + \delta^2 \left\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1 \right\|^2_{(L^2(\omega))^3} + \left\| \frac{\partial \mathcal{V}}{\partial s_1}\cdot\mathbf{Rt}_2 - \frac{\partial \mathcal{V}}{\partial s_2}\cdot\mathbf{Rt}_1 \right\|^2_{L^2(\omega)}.$$

Then we have

Proposition 1. *There exists a constant C such that for all $\mathbf{v} \in \mathbf{D}$*

$$\frac{1}{\delta^2}\|\overline{V}\|^2_{(L^2(\omega;H^1(-1,1)))^3} + \delta^2 \left\| \frac{\partial \mathbf{R}}{\partial s_\alpha} \right\|^2_{(L^2(\omega))^{3\times 3}} + \left\| \frac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{Rt}_\alpha \right\|^2_{(L^2(\omega))^3} \leqslant C\mathcal{E}_\delta(\mathbf{v}). \tag{4}$$

Now we assume that the shell is clamped on a part $\Gamma_{0,\delta} = \Phi(\gamma_0 \times]-\delta,\delta[)$ of the lateral boundary, where γ_0 is a nonempty open subset of $\partial\omega$. The set of admissible triples is then $\mathbf{D}_{\gamma_0} = \{\mathbf{v} = (\mathcal{V}, \mathbf{R}, \overline{V}) \in \mathbf{D} \mid \mathcal{V} = \phi,\ \mathbf{R} = \mathbf{I}_3 \text{ on } \gamma_0\}$ (see [1]). In some sense, the following corollary gives a Korn's inequality on $\mathbf{D}_{\gamma_0}$ with respect to the quantity $\mathcal{E}_\delta(\mathbf{v})$.

Corollary 2. *There exists a constant C such that for all $\mathbf{v} \in \mathbf{D}_{\gamma_0}$*

$$\|\mathcal{V} - \phi\|^2_{(H^1(\omega))^3} + \|\mathbf{R} - \mathbf{I}_3\|^2_{(H^1(\omega))^{3\times 3}} \leqslant \frac{C}{\delta^2}\mathcal{E}_\delta(\mathbf{v}). \tag{5}$$

4. Elastic shells

We consider an elastic shell submitted to applied body forces $f_{\kappa,\delta} \in (L^2(\mathcal{Q}_\delta))^3$ and whose total energy, defined over $\mathbf{U}_\delta = \{v \in (H^1(\mathcal{Q}_\delta))^3 \mid v = I_d \text{ on } \Gamma_{0,\delta}\}$, is given by

$$J_{\kappa,\delta}(v) = \int_{\mathcal{Q}_\delta} W(\nabla v) - \int_{\mathcal{Q}_\delta} f_{\kappa,\delta}\cdot(v - I_d), \quad \text{with } W(F) = \begin{cases} \widehat{W}(\frac{1}{2}(F^tF - \mathbf{I}_3)) & \text{if } \det(F) > 0, \\ +\infty & \text{if } \det(F) \leqslant 0 \end{cases}$$

and where $\widehat{W}$ is a continuous function defined on the set $\mathbf{S}_3$ of symmetric matrices such that (the reader is referred to [2,3] and [5])

$$\exists c > 0 \quad \text{such that } \forall E \in \mathbf{S}_3, \quad \widehat{W}(E) \geqslant c|||E|||^2, \tag{6}$$

$$\forall \varepsilon > 0, \quad \exists \theta > 0, \quad \text{such that } \forall E \in \mathbf{S}_3, \quad |||E||| \leqslant \theta \implies \big|\widehat{W}(E) - Q(E)\big| \leqslant \varepsilon |||E|||^2, \tag{7}$$

where Q is a positive quadratic form. In order to scale the forces $f_{\kappa,\delta}$, we assume that

$$f_{\kappa,\delta}(x) = \delta^{\kappa} f(s_1, s_2) + \delta^{\kappa-2} s_3 g(s_1, s_2) \quad \text{for a.e. } x \in \mathcal{Q}_\delta,\ f, g \in \left(L^2(\omega)\right)^3.$$

The Korn's inequalities established in [1], the assumption on the forces and (6) allow one to prove that there exists a constant c such that

$$c \leqslant \frac{m_{\kappa,\delta}}{\delta^{2\kappa-1}} = \frac{\inf_{v\in \mathbf{U}_\delta} J_{\kappa,\delta}(v)}{\delta^{2\kappa-1}} \leqslant 0.$$

5. The simplified elastic model for shells

According to the analysis of Section 3 and to assumption (7), we introduce the simplified total energy over the fixed domain Ω

$$J^s_{\kappa,\delta}(\mathbf{v}) = \delta \int_\Omega Q\left((\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-T} \widehat{E}_\delta(\mathbf{v}) (\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-1}\right) \det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\, ds_1\, ds_2\, dS_3$$
$$+ \delta^3 \left\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1 \right\|^2_{(L^2(\omega))^3} + \delta \left\| \frac{\partial \mathcal{V}}{\partial s_1}\cdot \mathbf{R}\mathbf{t}_2 - \frac{\partial \mathcal{V}}{\partial s_2}\cdot \mathbf{R}\mathbf{t}_1 \right\|^2_{L^2(\omega)} - \delta^{\kappa+1}\mathcal{L}(\mathcal{V}, \mathbf{R})$$

where

$$\mathcal{L}(\mathcal{V}, \mathbf{R}) = 2\int_\omega \left[f\cdot(\mathcal{V}-\phi) + \frac{1}{3} g\cdot(\mathbf{R}-\mathbf{I}_3)\mathbf{n} \right] \det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\, ds_1\, ds_2$$
$$+ \frac{2}{3}\int_\omega g\cdot(\mathcal{V}-\phi)\left[\det\left(\frac{\partial \mathbf{n}}{\partial s_1}\Big|\mathbf{t}_2\Big|\mathbf{n}\right) + \det\left(\mathbf{t}_1\Big|\frac{\partial \mathbf{n}}{\partial s_2}\Big|\mathbf{n}\right)\right] ds_1\, ds_2.$$

The following result is easy to establish:

Theorem 3. *There exists $\mathbf{v}_\delta \in \mathbf{D}_{\gamma_0}$ such that $J^s_{\kappa,\delta}(\mathbf{v}_\delta) = \min_{\mathbf{v}\in \mathbf{D}_{\gamma_0}} J^s_{\kappa,\delta}(\mathbf{v})$.*

Due to Corollary 2, (6) and (7), we get $\mathcal{E}_\delta(\mathbf{v}_\delta) \leqslant C\delta^{2(\kappa-1)}(\|f\|_{(L^2(\omega))^3} + \|g\|_{(L^2(\Omega))^3})^2$.
In order to compare the limit behavior of the two models, we set

$$m^s_{\kappa,\delta} = J^s_{\kappa,\delta}(\mathbf{v}_\delta) = \min_{\mathbf{v}\in \mathbf{D}_{\gamma_0}} J^s_{\kappa,\delta}(\mathbf{v}),$$

and the above estimate shows that there exists a constant independent of δ such that $c \leqslant \frac{m^s_{\kappa,\delta}}{\delta^{2\kappa-1}} \leqslant 0$.

6. Justification of the simplified model. Case $\kappa = 2$

The following result shows that the two energies introduced above have the same asymptotic behavior. In some specific cases, the "limit" energy $\mathcal{J}_2$ can be obtained through a Γ-convergence technique (see e.g. [5]).

Theorem 4. *We have $m_2 = \lim_{\delta\to 0} \frac{m_{2,\delta}}{\delta^3} = \lim_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3}$.*
Setting $\mathbf{D}_2 = \{\mathbf{v}\in \mathbf{D}_{\gamma_0} \mid \mathcal{V}\in (H^2(\omega))^3,\ \text{and}\ \frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha\}$ we have $m_2 = \min_{\mathbf{v}\in \mathbf{D}_2} \mathcal{J}_2(\mathbf{v})$ where

$$\mathcal{J}_2(\mathbf{v}) = \int_\Omega Q\left((\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-T} \widehat{\mathbf{E}}(\mathbf{v}) (\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-1}\right) \det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) - \mathcal{L}(\mathcal{V}, \mathbf{R})$$

with

$$\widehat{\mathbf{E}}(\mathbf{v}) = \begin{pmatrix} S_3 \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{R}\mathbf{t}_1 & S_3 \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{R}\mathbf{t}_2 & \frac{1}{2}\frac{\partial \overline{V}}{\partial S_3}\cdot \mathbf{t}_1 \\ * & S_3 \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot \mathbf{R}\mathbf{t}_2 & \frac{1}{2}\frac{\partial \overline{V}}{\partial s_3}\cdot \mathbf{t}_2 \\ * & * & \frac{\partial \overline{V}}{\partial s_3}\cdot \mathbf{n} \end{pmatrix}.$$

Idea of proof. Step 1. Proposition 1 and Corollary 2 together with $c \leqslant \frac{m^s_{2,\delta}}{\delta^3} \leqslant 0$ allow to obtain estimates on $\mathbf{v}_\delta$ in $\mathbf{D}_{\gamma_0}$. Then, proceeding as [1], we obtain (up to a subsequence) that the limit $\mathbf{v}$ of $\{\mathbf{v}_\delta\}$ belongs to $\mathbf{D}_2$ and we derive the weak limit of $\frac{1}{\delta}\Pi_\delta(\widehat{\mathbf{E}}_\delta(\mathbf{v}_\delta))$ in terms of $\mathbf{v}$ and the limit $\mathcal{Z}_{i\alpha}$ of $\frac{1}{\delta}\mathcal{Z}_{i\alpha,\delta}$. This gives a lower bound for $\liminf_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3}$ which depends on $\mathbf{v}$

and the $\mathcal{Z}_{i\alpha}$'s. Minimizing this lower bound w.r.t. the $\mathcal{Z}_{i\alpha}$'s leads to $m_2 \leqslant \liminf_{\delta\to 0} \frac{m^s_{2,\delta}}{\delta^3}$. Due to the definition of $\mathbf{D}_2$ we also have for any δ, $m_2 \geqslant \frac{m^s_{2,\delta}}{\delta^3}$. Then proving that m_2 is achieved on $\mathbf{D}_2$ is easy.

Step 2. Consider a sequence $\{v_\delta\}$ in $\mathbf{U}_\delta$ such that $\liminf_{\delta\to 0} \frac{m_{2,\delta}}{\delta^3} = \lim_{\delta\to 0} \frac{J_{2,\delta}(v_\delta)}{\delta^3}$. Using (6) and the Korn's inequalities (see [1]) we get

$$\big\| dist\big(\nabla_x v_\delta, SO(3)\big)\big\|_{L^2(\mathcal{Q}_\delta)} \leqslant C\delta^{3/2}, \qquad \left\| \frac{1}{2}\big\{\nabla_x v_\delta^T \nabla_x v_\delta - \mathbf{I}_3\big\}\right\|_{(L^2(\mathcal{Q}_\delta))^{3\times 3}} \leqslant C\delta^{3/2}.$$

The L^1-weak convergence (up to a subsequence) of the sequence $\frac{1}{\delta}\Pi_\delta(\frac{1}{2}\{\nabla_x v_\delta^T \nabla_x v_\delta - \mathbf{I}_3\})$ established in [1] is a L^2-weak convergence. The above estimates and (7) lead to $m_2 \leqslant \liminf_{\delta\to 0} \frac{m_{2,\delta}}{\delta^3}$.

Let us consider a minimizer $\mathbf{v}_0 = (\mathcal{V}_0, \mathbf{R}_0, \overline{V}_0) \in \mathbf{D}_2$ of $\mathcal{J}_2$, we construct a sequence $((\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{V}_\delta))_{\delta>0}$ of approximations such that the associated deformation v_δ belongs to $\mathbf{U}_\delta$ and satisfies $\det(\nabla_x v_\delta) > 0$ a.e., $\|(\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3\|_{(L^2(\Omega_\delta))^{3\times 3}} \leqslant C\delta^{3/2}$ and $\frac{1}{\sqrt{\delta}}\Pi_\delta(\nabla_x v_\delta - \mathbf{R}) \longrightarrow 0$ strongly in $(L^4(\Omega))^3$. Finally, using again (7) we prove that $\limsup_{\delta\to 0} \frac{J_{2,\delta}(v_\delta)}{\delta^3} \leqslant m_2$. $\square$

In the minimization problems in Theorems 3 and 4, we can eliminate $\overline{V}$. Let us give the resulting functional of $(\mathcal{V}, \mathbf{R})$ in the case of a plate and of a St-Venant–Kirchhoff's law for which we have

$$\widehat{W}(F) = \begin{cases} \frac{\lambda}{8}(tr(F^T F - \mathbf{I}_3))^2 + \frac{\mu}{4} tr((F^T F - \mathbf{I}_3)^2) & \text{if } \det(F) > 0, \\ +\infty & \text{if } \det(F) \leqslant 0. \end{cases}$$

We obtain $m^s_{2,\delta} = \min_{(\mathcal{V},\mathbf{R})\in \mathbf{V}} \mathcal{F}^s_\delta(\mathcal{V}, \mathbf{R})$ where $\mathbf{V} = \{(\mathcal{V}, \mathbf{R}) \in (H^1(\omega))^3 \times (H^1(\omega))^{3\times 3} \mid (\mathcal{V}, \mathbf{R}, 0) \in \mathbf{D}_{\gamma_0}\}$ and

$$\begin{aligned}\mathcal{F}^s_\delta(\mathcal{V}, \mathbf{R}) = \delta\Bigg[& \frac{E\delta^2}{3(1-\nu^2)} \int_\omega (1-\nu) \sum_{\alpha,\beta=1}^{2} \big(\Gamma_{\alpha\beta}(\mathbf{R})\big)^2 + \nu\big(\Gamma_{11}(\mathbf{R}) + \Gamma_{22}(\mathbf{R})\big)^2 \\ & + \frac{E}{(1-\nu^2)} \int_\omega (1-\nu) \sum_{\alpha,\beta=1}^{2} (\mathcal{Z}_{\alpha\beta})^2 + \nu(\mathcal{Z}_{11} + \mathcal{Z}_{22})^2 + \frac{5E}{12(1+\nu)} \int_\omega \big(\mathcal{Z}_{31}^2 + \mathcal{Z}_{32}^2\big) \\ & + \delta^2 \left\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1 \right\|^2_{(L^2(\omega))^3} + \left\| \frac{\partial \mathcal{V}}{\partial s_1}\cdot \mathbf{R}\mathbf{t}_2 - \frac{\partial \mathcal{V}}{\partial s_2}\cdot \mathbf{R}\mathbf{t}_1 \right\|^2_{L^2(\omega)}\Bigg] - \delta^3 \mathcal{L}(\mathcal{V}, \mathbf{R}).\end{aligned}$$

In the same way, we have $m_2 = \min_{(\mathcal{V},\mathbf{R})\in \mathbf{V}_2} \mathcal{F}(\mathcal{V}, \mathbf{R})$ where $\mathbf{V}_2 = \{(\mathcal{V}, \mathbf{R}) \in \mathbf{V} \mid (\mathcal{V}, \mathbf{R}, 0) \in \mathbf{D}_2\}$ and

$$\mathcal{F}(\mathcal{V}, \mathbf{R}) = \frac{E}{3(1-\nu^2)} \int_\omega \Bigg[(1-\nu) \sum_{\alpha,\beta=1}^{2} \big(\Gamma_{\alpha\beta}(\mathbf{R})\big)^2 + \nu\big(\Gamma_{11}(\mathbf{R}) + \Gamma_{22}(\mathbf{R})\big)^2\Bigg] - \mathcal{L}(\mathcal{V}, \mathbf{R}).$$

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