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The Ghirlanda–Guerra identities for mixed p-spin model

Les identités de Ghirlanda–Guerra pour les mélanges de modèles à p-spin

Dmitry Panchenko

Department of Mathematics, Texas A&M University, 77843 College Station, TX, USA

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ABSTRACT

We show that, under the conditions known to imply the validity of the Parisi formula, if the generic Sherrington–Kirkpatrick Hamiltonian contains a p-spin term then the Ghirlanda–Guerra identities for the pth power of the overlap hold in a strong sense without averaging. This implies strong version of the extended Ghirlanda–Guerra identities for mixed p-spin models than contain terms for all even $p \geqslant 2$ and $p = 1$.

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RÉSUMÉ

Nous montrons que sous les conditions connues pour impliquer la validité de la formule de Parisi, si l'Hamiltonien du modè le générique de Sherrington–Kirkpatrick Hamiltonien contient un « Hamiltonien de p-spin » alors les identités de Ghirlanda–Guerra pour la puissance p des recouvrements sont valides dans un sens fort (et pas seulement en moyenne sur les parametres).

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1. Introduction and main result

A generic mixed p-spin Hamiltonian $H_N(\boldsymbol{\sigma})$ indexed by spin configurations $\boldsymbol{\sigma} \in \{-1,+1\}^N$ is defined as a linear combination

$$H_N(\boldsymbol{\sigma}) = \sum_{p \geqslant 1} \beta_p H_p(\boldsymbol{\sigma}) \tag{1}$$

of p-spin Sherrington–Kirkpatrick Hamiltonians

$$H_p(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{1 \leqslant i_1,\dots,i_p \leqslant N} g_{i_1,\dots,i_p} \sigma_{i_1} \dots \sigma_{i_p} \tag{2}$$

where $(g_{i_1,\dots,i_p})$ are i.i.d. standard Gaussian random variables, also independent for all $p \geqslant 1$. For simplicity of notation, we will keep the dependence of H_p on N implicit. If a model involves an external field parameter $h \in \mathbb{R}$ then the (random) Gibbs measure on $\{-1,+1\}^N$ corresponding to the Hamiltonian H_N is defined by

E-mail address: panchenko.math@gmail.com.

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$$G_N(\boldsymbol{\sigma}) = \frac{1}{Z_N}\exp\left(H_N(\boldsymbol{\sigma}) + h\sum_{i\leqslant N}\sigma_i\right) \tag{3}$$

where Z_N is the normalizing factor called the partition function. As usual, we will denote by $\langle\cdot\rangle$ the expectation with respect to the product Gibbs measure $G_N^{\otimes\infty}$. One of the most important properties of the Gibbs measure G_N was discovered by Ghirlanda and Guerra in [2] who showed that on average over some small perturbation of the parameters (β_p) in (1) the annealed product Gibbs measure satisfies a family of distributional identities which are now called the Ghirlanda–Guerra identities. A more convenient version of this result proved in [7] can be formulated as follows. There exists a small perturbation $(\beta_{N,p})$ of the parameters in (1) such that all $\beta_{N,p}\to\beta_p$ and such that for all $p\geqslant 1$, $n\geqslant 2$ and any function $f = f(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n): (\{-1,+1\}^N)^n \to [-1,1]$,

$$\lim_{N\to\infty}\left|\mathbb{E}\langle f R_{1,n+1}^p\rangle - \frac{1}{n}\mathbb{E}\langle f\rangle\mathbb{E}\langle R_{1,2}^p\rangle - \frac{1}{n}\sum_{l=2}^{n}\mathbb{E}\langle f R_{1,l}^p\rangle\right| = 0 \tag{4}$$

where $\langle\cdot\rangle$ is now the Gibbs average corresponding to the Hamiltonian (1) with perturbed parameters $(\beta_{N,p})$ and $R_{l,l'} = N^{-1}\sum_{i\leqslant N}\sigma_i^l\sigma_i^{l'}$ is the overlap of configurations $\boldsymbol{\sigma}^l$ and $\boldsymbol{\sigma}^{l'}$. Of course, the ultimate goal would be to show that (4) holds without perturbing the parameters (β_p) which would mean that the joint distribution of the overlaps $(R_{l,l'})_{l,l'\geqslant 1}$ under the annealed product Gibbs measure $\mathbb{E}G_N^{\otimes\infty}$ asymptotically satisfies the following distributional identities (up to symmetry considerations): for any $n\geqslant 2$, conditionally on $(R_{l,l'})_{1\leqslant l<l'\leqslant n}$ the law of $R_{1,n+1}$ is given by the mixture $n^{-1}\mu + n^{-1}\sum_{l=2}^n \delta_{R_{1,l}}$ where μ is the law of $R_{1,2}$. Toward this goal, a stronger version of (4) for the original Hamiltonian (1) without any perturbation of parameters was proved for $p=1$ in [1] under the additional assumption that $\beta_1\neq 0$ and a non-restrictive assumption on the limit of the free energy $F_N = N^{-1}\mathbb{E}\log Z_N$. Here we will prove the same result for all p under the assumptions and as a direct consequence of the seminal work of Talagrand in [6] where the validity of the Parisi formula was proved. Namely, from now on we will assume that the sum in (1) is taken over $p=1$ and even $p\geqslant 2$. In this case, it was proved in [6] that the limit of the free energy

$$\lim_{N\to\infty} F_N(\boldsymbol{\beta}) = P(\boldsymbol{\beta}) \tag{5}$$

exists and is given by the Parisi formula $P(\boldsymbol{\beta})$ discovered in [4]. The exact definition of $P(\boldsymbol{\beta})$ will not be important to us and the only nontrivial property that we will use is its differentiability in each coordinate β_p which was proved in [5] (see also [3]).

Theorem 1.1. *For $p=1$ and for all even $p\geqslant 2$,*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}) - \mathbb{E}\langle H_p(\boldsymbol{\sigma})\rangle\big|\big\rangle = 0. \tag{6}$$

If $\beta_p\neq 0$ then (6) implies (4) by the usual integration by parts. In particular, if $\beta_p\neq 0$ for $p=1$ and all even $p\geqslant 2$, the positivity principle of Talagrand proved in [8] implies the strong version of the extended Ghirlanda–Guerra identities without any perturbation of the parameters.

Remark. We will see that the proof does not depend on the specific form of the Hamiltonian (1) and the result can be formulated in more generality. Namely, given a sequence of random measures ν_N on some measurable space (Σ, S) and a sequence of random processes A_N indexed by $\boldsymbol{\sigma}\in\Sigma$, consider a sequence of Gibbs' measures G_N defined by the change of density

$$dG_N(\boldsymbol{\sigma}) = Z_N^{-1}\exp\big(xA_N(\boldsymbol{\sigma})\big)\,\mathrm{d}\nu_N(\boldsymbol{\sigma}).$$

Let $\psi_N(x) = N^{-1}\log Z_N$ and $F_N(x) = \mathbb{E}\psi_N(x)$. Suppose that $\mathbb{E}|\psi_N(x) - F_N(x)|\to 0$ and $F_N(x)\to P(x)$ in some neighborhood of x_0, and suppose that the limit $P(x)$ is differentiable at x_0. Then

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\big\langle\big|A_N(\boldsymbol{\sigma}) - \mathbb{E}\langle A_N(\boldsymbol{\sigma})\rangle_{x_0}\big|\big\rangle_{x_0} = 0, \tag{7}$$

assuming some measurability and integrability conditions on A_N and ν_N which will be clear from the proof and are usually trivially satisfied. In Theorem 1.1 we simply appeal to the results in [5] and [6].

Proof of Theorem 1.1. It has been observed for a long time that (see, for example, [1])

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\big|\langle H_p(\boldsymbol{\sigma})\rangle - \mathbb{E}\langle H_p(\boldsymbol{\sigma})\rangle\big| = 0.$$

This is where one uses the fact that $\mathbb{E}|\psi_N - F_N| \to 0$, which is well known for p-spin models. It remains to prove that

$$\lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle\big|\big\rangle = 0. \tag{8}$$

This was proved in [1] for $p = 1$, but here we will show how this can be obtained as a direct consequence of (5) for all p. Let $\langle\cdot\rangle_x$ denote the Gibbs average corresponding to the Hamiltonian (1) where β_p has been replaced by x. Consider $\beta_p' > \beta_p$ and let $\delta = \beta_p' - \beta_p$. We start with the following obvious equation,

$$\int_{\beta_p}^{\beta_p'} \mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big\rangle_x\,dx = \delta\,\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big\rangle_{\beta_p} + \int_{\beta_p}^{\beta_p'}\int_{\beta_p}^{x} \frac{\partial}{\partial t}\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big\rangle_t\,dt\,dx. \tag{9}$$

Since

$$\begin{aligned}\left|\frac{\partial}{\partial t}\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big\rangle_t\right| &= \big|\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big(H_p(\boldsymbol{\sigma}^1) + H_p(\boldsymbol{\sigma}^2) - 2H_p(\boldsymbol{\sigma}^3)\big)\big\rangle_t\big| \\ &\leqslant 2\mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big)^2\big\rangle_t \leqslant 8\mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_t\big)^2\big\rangle_t\end{aligned}$$

Eq. (9) implies

$$\begin{aligned}\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big\rangle_{\beta_p} &\leqslant \frac{1}{\delta}\int_{\beta_p}^{\beta_p'} \mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big\rangle_x\,dx + \frac{8}{\delta}\int_{\beta_p}^{\beta_p'}\int_{\beta_p}^{x} \mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_t\big)^2\big\rangle_t\,dt\,dx \\ &\leqslant \frac{2}{\delta}\int_{\beta_p}^{\beta_p'} \mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_x\big|\big\rangle_x\,dx + 8\int_{\beta_p}^{\beta_p'} \mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_t\big)^2\big\rangle_t\,dt \\ &\leqslant 2\left(\frac{1}{\delta}\int_{\beta_p}^{\beta_p'} \mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_x\big)^2\big\rangle_x\,dx\right)^{1/2} + 8\int_{\beta_p}^{\beta_p'} \mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_x\big)^2\big\rangle_x\,dx.\end{aligned}$$

Therefore, if we denote

$$\Delta_N = \frac{1}{N}\int_{\beta_p}^{\beta_p'} \mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_x\big)^2\big\rangle_x\,dx$$

we showed that

$$\frac{1}{N}\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle\big|\big\rangle \leqslant \frac{1}{N}\,\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}^1) - H_p(\boldsymbol{\sigma}^2)\big|\big\rangle \leqslant 2\sqrt{\frac{\Delta_N}{N\delta}} + 8\Delta_N. \tag{10}$$

If for a moment we think of $F_N = F_N(x)$ as a function of x only then

$$F_N'(x) = \frac{1}{N}\mathbb{E}\big\langle H_p(\boldsymbol{\sigma})\big\rangle_x \quad\text{and}\quad F_N''(x) = \frac{1}{N}\mathbb{E}\big\langle\big(H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle_x\big)^2\big\rangle_x$$

so that $\Delta_N = F_N'(\beta_p') - F_N'(\beta_p)$. Since $F_N(x)$ is convex, for any $\gamma > 0$,

$$\Delta_N = F_N'\big(\beta_p'\big) - F_N'(\beta_p) \leqslant \frac{F_N(\beta_p' + \gamma) - F_N(\beta_p')}{\gamma} - \frac{F_N(\beta_p) - F_N(\beta_p - \gamma)}{\gamma}$$

and, therefore, Eqs. (10) and (5) now imply

$$\limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\big\langle\big|H_p(\boldsymbol{\sigma}) - \big\langle H_p(\boldsymbol{\sigma})\big\rangle\big|\big\rangle \leqslant 8\left(\frac{P(\beta_p' + \gamma) - P(\beta_p')}{\gamma} - \frac{P(\beta_p) - P(\beta_p - \gamma)}{\gamma}\right)$$

where again we write $P = P(x)$ as a function of x only. Letting $\beta_p' \to \beta_p$ first and then letting $\gamma \to 0$ and using that $P(x)$ is differentiable proves the result. □

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