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Partial Differential Equations/Numerical Analysis

On the determination of Dirichlet or transmission eigenvalues from far field data ☆

Sur la détermination des fréquences propres de Dirichlet ou de transmission à partir de l'opérateur de champs lointains

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A B S T R A C T

We show that the Herglotz wave function with kernel the Tikhonov regularized solution of the far field equation becomes unbounded as the regularization parameter tends to zero iff the wavenumber k belongs to a discrete set of values. When the scatterer is such that the total field vanishes on the boundary, these values correspond to the square root of Dirichlet eigenvalues for $-\Delta$. When the scatterer is a nonabsorbing inhomogeneous medium these values correspond to so-called transmission eigenvalues.

R É S U M É

Nous montrons qu'une certaine norme de l'onde de Herglotz ayant pour noyau la régularisée de Tikhonov de la solution de l'équation de champs lointains tend vers ∞ lorsque le paramètre de régularisation tend vers 0, si le nombre d'onde k appartient à un ensemble discret de valeurs. Lorsque l'objet diffractant est tel que l'onde s'annule sur sa frontière, ces valeurs sont les racines carrées des valeurs propres de Dirichlet pour $-\Delta$. Lorsque l'objet diffractant est un milieu pénétrable non absorbant, ces valeurs coincident avec les dites valeurs propres de transmission.

Version française abrégée

Considérons le problème de diffraction d'une onde plane $u^i(x,d) = e^{ikx\cdot d}$ ($k > 0$ désigne le nombre d'onde et d un vecteur unitaire) par une inclusion $D \subset \mathbb{R}^2$ (ouvert borné Lipschitzien de complément connexe). Nous nous intéressons aussi bien au cas où l'inclusion modélise un obstacle, auquel cas le champ total $u = u^i + u^s$ satisfait (1), qu'au cas où l'inclusion modélise un objet pénétrable d'indice n (tq. $|n-1| \geqslant \gamma > 0$ dans D), auquel cas $u = u^i + u^s$ satisfait (7). On note $u_\infty(\hat{x}, d)$ le champ lointain associé à u^s et défini par (2). On introduit l'opérateur de champs lointains $F : L^2(\Omega) \to L^2(\Omega)$ comme étant l'opérateur intégral de noyau u_∞ (voir (3)). En particulier, Fg est le champ lointain associé à $\mathcal{H}g(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds_d$, dite onde de Herglotz de noyau g. On note F^δ l'opérateur associé à des mesures bruitées des champs lointains et on suppose qu'il

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verifie (4). Nous allons montré qu'il est possible de repérer certaines fréquences caractéristiques de l'inclusion D, à partir des mesures de champs lointains, en se basant sur le comportement de la suite $(\mathcal{H}g_{z,\delta})_\delta$, où $g_{z,\delta}$ désigne le minimiseur de la fonctionnelle de Tikhonov (5) associée à un paramètre de régularisation $\epsilon(\delta) \to 0$ lorsque $\delta \to 0$. On suppose pour cela que (6) est vérifiée (ce qui est le cas par exemple lorsque F est d'image dense ; voir appendice).

Cas où D est un obstacle. Il est montré dans [1] que si $z \in D$ et si k^2 n'est pas une valeur propre de Dirichlet pour $-\Delta$ dans D, alors $\|\mathcal{H}g_{z,\delta}\|_{H^1(D)}$ reste bornée lorsque $\delta \to 0$. Nous complètons ce résultat en montrant (Theorem 2.1) que si k^2 est une valeur propre de Dirichlet pour $-\Delta$ dans D, $\|\mathcal{H}g_{z,\delta}\|_{H^1(D)}$ ne peut pas être bornée lorsque $\delta \to 0$ pour presque tout $z \in D$.

Cas où D est un objet pénétrable. Nous définissons dans ce cas les *fréquences de transmissions* comme étant les fréquences pour lesquelles le problème (8)–(9) admet une solution non triviale lorsque $\Phi = 0$. Nous montrons (Theorem 3.2) que pour ces fréquences, $\|\mathcal{H}g_{z,\delta}\|_{L^2(D)}$ ne peut pas être borné lorsque $\delta \to 0$ pour presque tout $z \in D$.

1. Introduction

The linear sampling method is probably the best known of the new class of qualitative methods that have recently been developed to solve time harmonic inverse scattering problems for acoustic and electromagnetic waves [3]. The solution of inverse scattering problems by the linear sampling method is based on solving an ill-posed far field equation by using Tikhonov regularization and since in general the far field equation has no solution this regularized solution does not converge as the regularization parameter tends to zero. However, Arens [1] was able to show for the case of scattering problem for the Helmholtz equation with Dirichlet boundary condition that, if the square of the wave number is not a Dirichlet eigenvalue for $-\Delta$ in D, the Herglotz wave function with the regularized solution of the far field equation as kernel converges in the $H^1(D)$ norm when the "sampling point" is in D. A similar result is also valid if the scattering object is a nonabsorbing inhomogeneous medium with support $\overline{D}$ and index of refraction n where now instead of requiring that the square of the wave number k is not a Dirichlet eigenvalue we must require that k is not a *transmission eigenvalue* (to be defined in Section 3) and $H^1(D)$ is replaced with $L^2(D)$.

In the above investigation the case when k^2 is a Dirichlet eigenvalue or k is a transmission eigenvalue was not addressed. However, it has recently been shown that in the case of scattering by a nonabsorbing inhomogeneous medium transmission eigenvalues can give valuable information on the index of refraction. In particular, a knowledge of the first transmission eigenvalue provides a lower bound for $\max_D n(x)$ [2,4,9]. The importance of these new results rests on the fact that numerical evidence indicates that transmission eigenvalues can be determined from a knowledge of the far field pattern of the scattered wave. In particular, if Tikhonov regularization is used to solve the far field equation (for the "sampling point" z inside the scattering object) and the L^2-norm of the regularized solution is plotted against the wave number k, then sharp peaks are seen at the values of k which, in the special case when transmission eigenvalues are known, correspond to these eigenvalues. However, a mathematical explanation that the location of these peaks corresponds to the location of transmission eigenvalues has yet to be established for general domains D and arbitrarily indices of refraction. The purpose of this paper is to provide this missing justification.

For the sake of presentation we limit ourselves to the two-dimensional case. In particular, the above problems correspond to the scattering of either acoustic or electromagnetic waves scattered by an infinite cylinder [3]. Everything in the following analysis holds true in the corresponding three-dimensional scalar case as well.

2. Scattering by an obstacle

We consider the scattering problem for the Helmholtz equation with Dirichlet boundary condition on the boundary of the scattering obstacle D. In particular, we assume that $D \subset \mathbb{R}^2$ is an open, bounded region with Lipschitz boundary ∂D such that $\mathbb{R}^2 \setminus \overline{D}$ is connected. Then, factoring out a term of the form $e^{-i\omega t}$ where ω is the frequency, the total field, $u = u^i + u^s$ satisfies the exterior boundary value problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \qquad u = 0 \quad \text{on } \partial D \quad \text{and} \quad \lim_{r\to\infty} \sqrt{r}\big(\partial u^s/\partial r - iku^s\big) = 0, \tag{1}$$

where the incident field u^i is given by $u^i(x,d) = e^{ikx\cdot d}$, $k > 0$, is the wave number, d is a unit vector, u^s is the scattered field and the Sömmerfeld radiation condition holds uniformly in $\hat{x} = x/|x|$ with $r = |x|$. It is shown in [3] that (1) has a unique solution in $u \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ and u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\big(r^{-3/2}\big) \tag{2}$$

as $r \to \infty$ where u_∞ is the *far field pattern* of the scattered wave. The far field pattern can now be used to define the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, \mathrm{d}s_d \tag{3}$$

for $g \in L^2(\Omega)$ where $\Omega := \{x \in \mathbb{R}^2\colon\ |x| = 1\}$. In particular, $(Fg)(\hat{x})$ is the far field pattern for the scattered field in (1) where u^i is the *Herglotz wave function* defined by $\mathcal{H}g(x) := \int_\Omega e^{ikx\cdot d} g(d)\,\mathrm{d}s_d$. Hence, $F = \mathcal{B}\mathcal{H}$ where $\mathcal{B}$ denotes the extension by continuity of the mapping $u^i(\cdot,d)|_{\partial D} \to u_\infty(\cdot,d)$ from $H^{1/2}(\partial D)$ into $L^2(\Omega)$. Let F^δ be an operator corresponding to noisy measurements $u^\delta_\infty(\hat{x},d)$: we call this operator the noisy operator. We assume that

$$F^\delta = \mathcal{B}^\delta \mathcal{H}, \quad \text{where } \left\|\mathcal{B}^\delta - \mathcal{B}\right\| \leqslant \delta \tag{4}$$

where $\delta > 0$ is a measure of the noise level and $\mathcal{B}^\delta$ denotes the noisy bounded linear operator associated with $\mathcal{B}$. In particular $F^\delta : L^2(\Omega) \to L^2(\Omega)$ is a bounded and compact linear operator.

Of particular interest to us in the sequel is the Tikhonov regularized solution $g^\delta_{z,\epsilon}$ of the far field equation defined as the unique minimizer of the *Tikhonov functional* [3],

$$\left\|F^\delta g - \Phi_\infty(\,\cdot\,,z)\right\|^2_{L^2(\Omega)} + \epsilon\|g\|^2_{L^2(\Omega)} \tag{5}$$

where the positive number ϵ is known as the *Tikhonov regularization parameter*, and where $\Phi_\infty(\hat{x},z) := \exp(-ik\hat{x}\cdot z + i\pi/4)/\sqrt{8\pi k}$ is the far field pattern of the fundamental solution $\Phi(x,z) := (i/4)H^{(1)}_0(k|x-z|)$, with $H^{(1)}_0$ denoting the Hankel function of the first kind of order zero. This regularized solution is used by the linear sampling method to determine ∂D from a knowledge of u^δ_∞.

Let $\epsilon(\delta)$ be a sequence of regularization parameters such that $\epsilon(\delta) \to 0$ as $\delta \to 0$ and let $g_{z,\delta} := g^\delta_{z,\epsilon(\delta)}$ be the minimizer of (5) with $\epsilon = \epsilon(\delta)$. It was shown by Arens [1] that if k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D and if $z \in D$, then $\mathcal{H}g_{z,\delta}$ converges in the $H^1(D)$ norm as $\delta \to 0$. We shall prove in the following that this is not in general true if k^2 is a Dirichlet eigenvalue. To this end we assume that the perturbed operator F^δ is such that for all points $z \in D$,

$$\lim_{\delta\to 0}\left\|F^\delta g_{z,\delta} - \Phi(\cdot,z)\right\|_{L^2(\Omega)} = 0. \tag{6}$$

We note that the equation $Fg = \Phi_\infty$ has in general no solution (and therefore the sequence $(g_{z,\delta})$ is in general not bounded as $\delta \to 0$). However, (6) is verified as soon as the operator F has dense range (see Appendix A), which is true except for the exceptional cases where k^2 is a Dirichlet eigenvalue for D associated with an eigenfunction that can be represented as a Herglotz wave function. We shall indicate in Appendix A the needed properties so that (6) holds even for these exceptional cases.

Theorem 2.1. *Let k^2 be a Dirichlet eigenvalue and assume that* (6) *is true. Then for almost every z in D $\|\mathcal{H}g_{z,\delta}\|_{H^1(D)}$ cannot be bounded as $\delta \to 0$.*

Proof. Assume that for a set of points $z \in D$ which has a positive measure, $\|\mathcal{H}g_{z,\delta}\|_{H^1(D)} \leqslant M$ for some constant $M > 0$ (the constant M may depend on z). From the estimate

$$\left\|F^\delta g_{z,\delta} - Fg_{z,\delta}\right\| \leqslant \left\|\mathcal{B}^\delta - \mathcal{B}\right\|\|\mathcal{H}g_{z,\delta}\|_{H^{1/2}(\partial D)}$$

and (6) one easily deduces that $\|Fg_{z,\delta} - \Phi_\infty(\cdot,z)\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$. On the other hand, there exists a subsequence of $v_n = \mathcal{H}g_{z,\delta_n}$ which converges weakly to some $v \in H^1(D)$ that satisfies $\Delta v + k^2 v = 0$ in D. By the compactness of the operator $\mathcal{B}$, we conclude that $\|B(v_n|_{\partial D}) - B(v|_{\partial D})\|_{L^2(\Omega)} = \|Fg_{z,\delta_n} - B(v|_{\partial D})\|_{L^2(\Omega)} \to 0$ as $\delta_n \to 0$. Therefore $B(v|_{\partial D}) = \Phi_\infty(\cdot\,,z)$ and from Rellich's lemma and the unique continuation principle $v = \Phi(\cdot\,,z)$ on ∂D. Since is k^2 is a Dirichlet eigenvalue, we conclude from [6, Theorem 53] that this is possible only if $w(z) := \int_{\partial D} \partial\varphi/\partial\nu\,\Phi(\cdot,z)\,\mathrm{d}s = 0$ where φ is a Dirichlet eigenfunction for $-\Delta$ in D associated with k^2. Then $w(z) = 0$ for $z \in D$ by unique continuation. On the other hand w is a radiating solution to the Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$ such that w vanishes on ∂D. The latter comes from the continuity of the single layer potential. Hence $w(z) = 0$ in $\mathbb{R}^2 \setminus \overline{D}$ and by the discontinuity property of the normal derivative of the single layer potential one deduces that $\partial\varphi/\partial\nu = 0$ on ∂D. By Green's representation formula we now have that $\varphi(x) = 0$ for $x \in D$, a contradiction since φ is an eigenvalue. The theorem now follows. □

3. Scattering by an inhomogeneous medium

We now turn our attention to the scattering problem for a dielectric inhomogeneous medium. This can be formulated as the problem of finding a function $u \in H^1_{loc}(\mathbb{R}^2)$ such that

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^2, \tag{7}$$

$u = u^s + u^i$ where again $u^i(x) = e^{ikx\cdot d}$ and u^s is the scattered field satisfying the Sömmerfeld radiation condition. The index of refraction n is assumed to be a real valued bounded function such that $n(x) = 1$ for $x \in \mathbb{R}^2 \setminus \overline{D}$ where D is a bounded domain having the same properties as in Section 2. Under our assumptions on D and n the scattering problem is uniquely solvable in $H^1_{loc}(\mathbb{R}^2)$. Furthermore, the scattered field u^s again has the asymptotic behavior (2). We also consider the far field operator F given by (3) as well as the same setting and hypothesis for the perturbed operator F^δ where now the operator $\mathcal{B}$

denotes the extension by continuity of the mapping $u^i(\cdot,d)|_D \to u_\infty(\cdot,d)$ from $H_{\mathrm{inc}}(D) := \{v \in L^2(D);\ \Delta v + k^2 v = 0 \text{ in } D\}$ into $L^2(\Omega)$. Considering the minimizer $g_{z,\delta} := g^\delta_{z,\epsilon(\delta)}$ for $z \in D$ we now investigate the behavior of $\|\mathcal{H} g_{z,\delta}\|_{L^2(D)}$ as $\delta \to 0$. This question is closely linked to the analysis of so-called *interior transmission problem* which looks for functions w_z and v_z in $L^2(D)$ such that $w_z - v_z \in H^2(D)$ and

$$\Delta w_z + k^2 n(x) w_z = 0 \quad \text{and} \quad \Delta v_z + k^2 v_z = 0 \quad \text{in } D, \tag{8}$$

$$w_z - v_z = \Phi(\cdot,z) \quad \text{and} \quad \frac{\partial w_z}{\partial \nu} - \frac{\partial v_z}{\partial \nu} = \frac{\partial \Phi(\cdot,z)}{\partial \nu} \quad \text{on } \partial D. \tag{9}$$

In particular $\mathcal{B} v_z = \Phi_\infty(\cdot,z)$ if and only if w_z and v_z satisfy (8)–(9). In the following, the *homogeneous interior transmission* problem, i.e. (8)–(9) with $\Phi(\cdot,z) = 0$ will play the same role as the Dirichlet eigenvalue problem plays in the analysis of Section 2. Motivated by potential applications in nondestructive testing [2], inside D we allow the possibility of regions $D_0 \subset D$ where the index of refraction is equal to one, i.e. D_0 is a cavity. More precisely, we consider a region $D_0 \subset D$, which can possibly be multiply connected, such that $\mathbb{R}^2 \setminus \overline{D}_0$ is connected with Lipschitz boundary ∂D_0 and assume that $n(x) = 1$ in D_0. We further assume that $|n-1|$ is bounded away from zero in $D \setminus \overline{D}_0$ so that $1/(n-1) \in L^\infty(D \setminus \overline{D}_0)$. The interior transmission problem (8)–(9) for this case has recently been investigated in [4] (see [9,5] and [10] for the case where $D_0 = \emptyset$). Let $\theta_z \in H^2(D)$ be the lifting function [8] such that $\theta_z = \Phi(\cdot,z)$ and $\partial\theta_z/\partial\nu = \partial\Phi(\cdot,z)/\partial\nu$ on ∂D. With the help of a cutoff function we can guarantee that $\theta_z = 0$ in D_θ such that $D_0 \subset D_\theta \subset D$. Then, one easily shows that $u_z := w_z - v_z - \theta_z \in V_0(D)$ where

$$V_0(D) := \left\{ u \in H^2(D) \text{ such that } u = 0,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D, \text{ and } \Delta u + k^2 u = 0 \text{ in } D_0 \right\}$$

and u_z satisfies the variational problem

$$\int_{D \setminus D_0} \frac{1}{n-1} \left\{ \left(\Delta + k^2 n\right)(u_z - \theta_z) \right\} \left\{ \left(\Delta + k^2\right)\varphi \right\} \mathrm{d}x = 0 \quad \text{for all } \varphi \in V_0(D). \tag{10}$$

In [4] it is shown that the Fredholm alternative holds for (10). We remark that in the case when $D_0 = \emptyset$ (i.e. there are no cavities inside the medium) everything in this section hold true: in this case $D \setminus \overline{D}_0$ reduces to D and $V_0(D)$ to $H_0^2(D)$ [10].

Definition 3.1. Values of $k > 0$ for which the homogeneous interior transmission problem (8)–(9) (i.e. in (9) the boundary data is 0) has nontrivial solutions are called transmission eigenvalues. The corresponding nontrivial solutions (w_0, v_0) are called transmission eigenfunctions.

If k is a transmission eigenvalue associated with (w_0, v_0) then $u_0 = w_0 - v_0$ is a nontrivial solution to (10) with $\theta_z = 0$. A major result of [4] (or [10] when $D_0 = \emptyset$) is that (10) can be written in the form $(I + B)u_z = f_{\theta_z}$, where $B : V_0(D) \to V_0(D)$ is a compact, self-adjoint operator (depending on k) and $f_{\theta_z} \in V_0(D)$. Moreover, it is shown that the solution exists except for a set of values k which is at most discrete. Thus the set of transmission eigenvalues forms at most a discrete set.

As stated before, we are again interested in the Tikhonov regularized solution $g_{z,\delta} := g^\delta_{z,\epsilon(\delta)}$ of (5) with $\epsilon = \epsilon(\delta) \to 0$ as $\delta \to 0$. We first note that the result of Arens for obstacle scattering [1] can be carried through for the case of inhomogeneous medium with real valued index of refraction. This follows from the fact that in this case the far field operator is normal and $\Phi(\cdot, z)$ is in the range of $(F^*F)^{1/4}$ for $z \in D$ [7]. In particular, it can be shown that if k is not a transmission eigenvalue then the $\mathcal{H} g_{z,\delta}$ converges in the $L^2(D)$ norm as $\delta \to 0$. The following theorem shows that this is not in general true if k is a transmission eigenvalue and (6) holds. Again this hypothesis is verified as soon as the operator F has dense range. In the present case, the latter is true except for the exceptional cases where k is a transmission eigenvalue associated with (w_0, v_0) such that v_0 can be represented as a Herglotz wave function.

Theorem 3.2. *Let k be a transmission eigenvalue and assume that* (6) *is true. Further assume that k^2 is not both a Dirichlet and Neumann eigenvalue of $-\Delta$ in D_0. Then for almost every $z \in D$ $\|\mathcal{H} g_{z,\delta}\|_{L^2(D)}$ cannot be bounded as $\delta \to 0$.*

Proof. Assume that for a set of points $z \in D$ which has a positive measure, $\|\mathcal{H} g_{z,\delta}\|_{H^1(D)} \leqslant M$ for some constant $M > 0$ (the constant M may depend on z). Then following the same arguments as in the proof of Theorem 2.1 one deduces the existence of $v_z \in H_{\mathrm{inc}}(D)$ such that $\mathcal{B}(v_z) = \Phi_\infty(\cdot,z)$. We can now deduce the existence of a solution v_z and w_z to (8)–(9) and hence a solution u_z to (10). Since k is a transmission eigenvalue, we have that the kernel of the corresponding operator $I + B$ (defined after Definition 3.1) is nontrivial and consists of all transmission eigenfunctions u_0. Hence, from the Fredholm alternative and the fact that B is self-adjoint, we have that

$$\int_{D \setminus \overline{D}_0} \frac{1}{n-1} \left(\Delta + k^2 n\right)\theta_z \left(\Delta + k^2\right) u_0 \,\mathrm{d}x = 0. \tag{11}$$

Integrating by parts in (11) and using the equation and the zero boundary conditions for u_0 as well as the definition of θ_z we obtain that

$$\int_{\partial D} \frac{1}{n-1}(\Delta + k^2 n)u_0 \frac{\partial \Phi(\cdot, z)}{\partial \nu}\,\mathrm{d}s - \int_{\partial D} \frac{\partial}{\partial \nu}\left(\frac{1}{n-1}(\Delta + k^2 n)u_0\right)\Phi(\cdot, z)\,\mathrm{d}s = 0, \tag{12}$$

where these integrals have to be understood in the sense of $H^{\mp 1/2}$ (resp. $H^{\mp 3/2}$) duality pairing. Defining $\psi(x) := \frac{1}{n-1}(\Delta + k^2 n(x))u_0(x)$ in $D \setminus \overline{D}_0$ then, since k^2 is not a Dirichlet and Neumann eigenvalue for $-\Delta$ in D_0, ψ can be extended to an $L^2(D)$ function satisfying $(\Delta + k^2)\psi(x) = 0$ in D (cf. [4]). Classical interior elliptic regularity results and the Green's representation theorem imply that

$$\psi(z) = \int_{\partial D} \left(\psi(x)\frac{\partial \Phi(x, z)}{\partial \nu} - \frac{\partial \psi(x)}{\partial \nu}\Phi(x, z)\right)\mathrm{d}s_x \quad \text{for } z \in D. \tag{13}$$

Eq. (12) and the unique continuation principle now show that $\psi = 0$ in D. Therefore $(\Delta + k^2 n(x))u_0(x) = 0$ in $D \setminus \overline{D}_0$. Since $u_0 \in V_0(D)$ one deduces from Green's representation theorem that $u_0 = 0$ in D, which is a contradiction. □

Appendix A

Assumption (6) is central in proving Theorems 2.1 and 3.2. We now give examples when this assumption is valid independently from the scattering problem. For the sake of clarity we shall keep the same notation as in Section 2 and since the result is independent of z we set $v_\infty = \Phi_\infty(\cdot, z)$ and $g_\delta = g_{z,\delta}$.

Lemma A.1. *If F has dense range then* (6) *is true.*

Proof. This follows from the fact that for all $g \in L^2(\Omega)$

$$\left\|F^\delta g_\delta - v_\infty\right\|^2_{L^2(\Omega)} \leqslant \left\|F^\delta g - v_\infty\right\|^2_{L^2(\Omega)} + \epsilon(\delta)\|g\|^2_{L^2(\Omega)} \leqslant \left\|Fg - v_\infty\right\|^2_{L^2(\Omega)} + \left(\epsilon(\delta) + \delta^2\|\mathcal{H}\|^2\right)\|g\|^2_{L^2(\Omega)}.$$

The first term in the upper bound can be made arbitrarily small by choosing appropriate g and the second one goes to zero as $\delta \to 0$ for fixed g. □

Let $\{u_i^\delta, \sigma_i^\delta, v_i^\delta, i \geqslant 1\}$ be a singular value decomposition of F^δ such that $F^\delta u_i^\delta = \sigma_i^\delta v_i^\delta$ and $\sigma_i^\delta > 0$.

Lemma A.2. *Property* (6) *is true if F^δ has dense range and the sequence (F^δ) satisfies*

1. *for all $v \in L^2(\Omega)$, $\sum_{i \geqslant N}(v, v_i^\delta)^2_{L^2} \to 0$ as $N \to \infty$ uniformly with respect to δ,*
2. *$\sqrt{\epsilon(\delta)}/\sigma_i^\delta \to 0$ as $\delta \to 0$ for all $i \geqslant 1$.*

Proof. Easy calculations show that

$$\left\|F^\delta g_\delta - v_\infty\right\|^2_{L^2(\Omega)} = \sum_{i \geqslant 1}\left(\frac{\epsilon(\delta)}{\epsilon(\delta) + (\sigma_i^\delta)^2}\right)^2\left(v_\infty, v_i^\delta\right)^2_{L^2} \leqslant \sum_{i < N}\left(\frac{\epsilon(\delta)}{\epsilon(\delta) + (\sigma_i^\delta)^2}\right)^2\left(v_\infty, v_i^\delta\right)^2_{L^2} + \sum_{i \geqslant N}\left(v_\infty, v_i^\delta\right)^2_{L^2}$$

where the second term in the upper bound can be made arbitrarily small by choosing N sufficiently large independently from δ and the first one goes to zero as $\delta \to 0$ for fixed N. □

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