C. R. Acad. Sci. Paris, Ser. I 348 (2010) 359–362

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Homological Algebra/Algebraic Geometry

Noncommutative Batalin–Vilkovisky geometry and matrix integrals ☆

Géométrie de Batalin–Vilkovisky non-commutative et intégrales matricielles

Serguei Barannikov

École normale superieure, 45, rue d'Ulm, 75230 Paris, France

ARTICLE INFO

Article history:
Received 17 May 2009
Accepted 22 January 2010
Available online 1 March 2010

Presented by Maxime Kontsevitch

ABSTRACT

I study the new type of *supersymmetric* matrix models associated with any solution to the quantum master equation of the noncommutative Batalin–Vilkovisky geometry. The asymptotic expansion of the matrix integrals gives homology classes in the Kontsevich compactification of the moduli spaces, which I associated with the solutions to the quantum master equation in my previous paper. I associate with the Bernstein–Leites matrix superalgebra equipped with an odd differentiation, whose square is *nonzero*, the family of cohomology classes of the compactification. This family is the generating function for the products of the tautological classes. The simplest example of my matrix integrals in the case of dimension zero is a supersymmetric extension of the Kontsevich model of 2-dimensional gravity.

RÉSUMÉ

J'associe un nouveau type d'intégrales matricielles super-symétriques avec une solution arbitraire de l'équation noncommutative de Batalin–Vilkovisky. Le cas le plus simple est une extension super-symétrique du modèle de Kontsevich de la gravité 2-dimensionnelle.

Notations

I work in the tensor category of super vector spaces, over an algebraically closed field k, $char(k) = 0$. Let $V = V^{even} \oplus V^{odd}$ be a super vector space. I denote by $\bar{\alpha}$ the parity of an element α and by ΠV the super vector space with inversed parity. For a finite group G acting on a vector space U, I denote U^G the space of invariants with respect to the action of G. Element $(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$ of $A^{\otimes n}$ is denoted by $(a_1, a_2, \ldots, a_n)$. Cyclic words, i.e. elements of the subspace $(V^{\otimes n})^{\mathbb{Z}/n\mathbb{Z}}$ are denoted $(a_1 \cdots a_n)^c$.

1. Noncommutative Batalin–Vilkovisky geometry

1.1. Even inner products

Let $B : V^{\otimes 2} \to k$ be an even symmetric inner product on V:

$$B(x, y) = (-1)^{\bar{x}\bar{y}} B(y, x).$$

☆ Preprint NI06043 (25/09/2006), Isaac Newton Institute for Mathematical Sciences. Preprint HAL, the CNRS electronic archive, 00102085 (28/09/2006).
E-mail addresses: sergueibar@gmail.com, serguei.barannikov@ens.fr.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.02.002

I introduced in [1] the space $F=\bigoplus_{n=1}^{n=\infty} F_n$

$$F_n=\left((\Pi V)^{\otimes n}\otimes k[\mathbb{S}_n]'\right)^{\mathbb{S}_n}, \tag{1}$$

where $k[\mathbb{S}_n]'$ denotes the super k-vector space with the basis indexed by elements (σ,ρ_σ), where $\sigma\in\mathbb{S}_n$ is a permutation with i_σ cycles σ_α and $\rho_\sigma=\sigma_1\wedge\cdots\wedge\sigma_{i_\sigma}$, $\rho_\sigma\in Det(Cycle(\sigma))$, $Det(Cycle(\sigma))=Symm^{i_\sigma}(k^{0|i_\sigma})$, is one of the generators of the one-dimensional determinant of the set of cycles of σ, i.e. ρ_σ is an order on the set of cycles defined up to even reordering, and $(\sigma,-\rho_\sigma)=-(\sigma,\rho_\sigma)$. The group $\mathbb{S}_n$ acts on $k[\mathbb{S}_n]'$ by conjugation. The space F is naturally isomorphic to:

$$F=Symm\left(\bigoplus_{j=1}^{\infty}\Pi\left(\Pi V^{\otimes j}\right)^{\mathbb{Z}/j\mathbb{Z}}\right).$$

The space F carries the naturally defined Batalin–Vilkovisky differential Δ (see [1] and references therein). It is the operator of the second order with respect to the multiplication and it is completely determined by its action on the second power of $\bigoplus_{j=1}^{\infty}\Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$. If one chooses a basis $\{a_i\}$ in ΠV, in which the antisymmetric even inner product defined by B on ΠV has the form $(-1)^{\bar a_i}B(\Pi a_i,\Pi a_j)=b_{ij}$, then the operator Δ sends a product of two cyclic words $(a_{\rho_1}\cdots a_{\rho_r})^c(a_{\tau_1}\cdots a_{\tau_t})^c$, to

$$\begin{aligned}
&\sum_{p,q}(-1)^{\varepsilon_1}b_{\rho_p\tau_q}(a_{\rho_1}\cdots a_{\rho_{p-1}}a_{\tau_{q+1}}\cdots a_{\tau_{q-1}}a_{\rho_{p+1}}\cdots a_{\rho_r})^c\\
&\quad+\sum_{p\pm1\neq q \bmod r}(-1)^{\varepsilon_2}b_{\rho_p\rho_q}(a_{\rho_1}\cdots a_{\rho_{p-1}}a_{\rho_{q+1}}\cdots a_{\rho_r})^c(a_{\rho_{p+1}}\cdots a_{\rho_{q-1}})^c(a_{\tau_1}\cdots a_{\tau_t})^c\\
&\quad+\sum_{p\pm1\neq q \bmod r}(-1)^{\varepsilon_3}b_{\tau_p\tau_q}(a_{\rho_1}\cdots a_{\rho_r})^c(a_{\tau_1}\cdots a_{\tau_{p-1}}a_{\tau_{q+1}}\cdots a_{\tau_t})^c(a_{\tau_{p+1}}\cdots a_{\tau_{q-1}})^c,
\end{aligned} \tag{2}$$

where ε_i are the standard Koszul signs, which take into the account that the parity of any cycle is opposite to the sum of parities of a_i: $\overline{(a_{\rho_1}\cdots a_{\rho_r})^c}=1+\sum\overline{a_{\rho_i}}$. It follows from [1, Proposition 2] that Δ defines a structure of Batalin–Vilkovisky algebra on F, in particular $\Delta^2=0$. The solutions of *the quantum master equation* in F

$$\hbar\Delta S+\frac{1}{2}\{S,S\}=0,\quad S=\sum_{g\geqslant 0}\hbar^{2g-1+i}S_{g,i,n},\quad S_{g,i,n}\in Symm^i\cap F_n^{even}, \tag{3}$$

with $S_{0,1,1}=0$, are in one-to-one correspondence, by [1, Theorem 1], with the structure of $\mathbb{Z}/2\mathbb{Z}$-graded *quantum A_∞-algebra* on V, i.e. the algebra over the $\mathbb{Z}/2\mathbb{Z}$-graded modular operad $\mathcal{F}_{\mathcal{K}}\mathbb{S}$, where $\mathbb{S}$ is the $\mathbb{Z}/2\mathbb{Z}$-graded version of the twisted modular $\mathcal{K}$-operad $\tilde{\mathfrak{s}}\Sigma\mathbb{S}[t]$, with components $k[\mathbb{S}'_n][t]$, described in [1]. The $(g=0,i=1)$-part is the cyclic $\mathbb{Z}/2\mathbb{Z}$-graded A_∞-algebra with the *even* invariant inner product on $Hom(V,k)\overset{B}{\simeq}V$. Recall, see [1], that for any solution to (3), with $S_{0,1,1}=S_{0,1,2}=0$, the value of partition function $c_S(G)$ on a stable ribbon graph G, with no legs, is defined by contracting the product of tensors $\bigotimes_{v\in Vert(G)}S_{g(v),i(v),n(v)}$ with $B^{\otimes Edge(G)}$ with appropriate signs.

Proposition 1. *(See [1, Sections 10, 11].) The graph complex $\mathcal{F}_{\mathcal{K}}\mathbb{S}((0,\gamma,\nu))$ (part of $\mathcal{F}_{\mathcal{K}}\mathbb{S}$ with no legs) is naturally identified with the cochain CW-complex $C^*(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu)$ of the Kontsevich compactification of the moduli spaces of Riemann surfaces from [7]. For any solution to the quantum master equation (3) in F, with $S_{0,1,1}=S_{0,1,2}=0$, the partition function on stable ribbon graphs $c_S(G)$ defines the characteristic homology class in $H_*(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu)$.*

1.2. Odd inner products

Let B be an odd symmetric inner product, $B:V^{\otimes 2}\to\Pi k$, $B(x,y)=(-1)^{\bar x\bar y}B(y,x)$. The analog of the space F in this situation has components

$$\widetilde{F}_n=\left(V^{\otimes n}\otimes k[\mathbb{S}_n]\right)^{\mathbb{S}_n},$$

where $k[\mathbb{S}_n]$ is the group algebra of $\mathbb{S}_n$, and $\mathbb{S}_n$ acts on $k[\mathbb{S}_n]$ by conjugation. The space $\widetilde{F}$ is naturally isomorphic in this case to:

$$\widetilde{F}=Symm\left(\bigoplus_{j=1}^{\infty}\left(V^{\otimes j}\right)^{\mathbb{Z}/j\mathbb{Z}}\right). \tag{4}$$

The space $\widetilde{F}$ carries the naturally defined second order differential defined by formula (2) with $a_i\in V$, $b_{ij}=B(a_i,a_j)$ and the Koszul signs ε_i, which now correspond to the standard parity of cycles: $\overline{(a_{\rho_1}\cdots a_{\rho_r})^c}=\sum\overline{a_{\rho_i}}$. Again, it follows from [1, Proposition 2], that $\Delta^2=0$, and that Δ defines the structure of Batalin–Vilkovisky algebra on $\widetilde{F}$.

The solutions of quantum master equation (3) in $\widetilde{F}$, with $S_{0,1,1}=0$, are in one-to-one correspondence, by [1], with the structure of algebra over the twisted modular operad $\mathcal{F}\widetilde{\mathbb{S}}$ on the vector space V. Here $\widetilde{\mathbb{S}}$ is the *untwisted* $\mathbb{Z}/2\mathbb{Z}$-graded version of $\mathfrak{s}\Sigma\mathbb{S}[t]$. The components $\widetilde{\mathbb{S}}((n))$ are the spaces $k[\mathbb{S}_n][t]$, with the composition maps defined as in [1, Section 9]. The Feynman transform $\mathcal{F}\widetilde{\mathbb{S}}$ is a $\mathcal{K}$-twisted modular operad, whose $(g=i=0)$-part corresponds to the cyclic A_∞-algebra with the *odd* invariant inner product on $Hom(\Pi V,k)\overset{B}{\simeq}V$.

Proposition 2. *(See [1, Sections 10, 11].) The graph complexes $\mathcal{F}\widetilde{\mathbb{S}}((0,\gamma,\nu))$ (part of $\mathcal{F}\widetilde{\mathbb{S}}$ with no legs) are naturally identified with the cochain CW-complexes $C^*(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu,\mathcal{L})$ of the Kontsevich compactification of the moduli spaces of Riemann surfaces with coefficients in the local system $\mathcal{L}=Det(P_\Sigma)$, where P_Σ is the set of marked points. For any solution to the quantum master equation* (3) *in $\widetilde{F}$, with $S_{0,1,1}=S_{0,1,2}=0$, the partition function on stable ribbon graphs $c_S(G)$ defines the characteristic homology class in $H_*(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu,\mathcal{L})$.*

2. Supersymmetric matrix integrals

2.1. Odd inner product

Let $S(a_i)$ be a solution to the quantum master equation (3) from $\widetilde{F}$, with $S_{0,1,2}=S_{0,1,1}=0$. Consider the vector space

$$M=Hom\big(V,End(U)\big)$$

where $\dim U=(d|d)$. The supertrace functional on $End(U)$ gives a natural extension to M of the odd symmetric inner product on $Hom(V,k)$ dual to B. Let us extend S to a function S_{gl} on M so that each cyclically symmetric tensor goes to the supertrace of the product of the corresponding matrices from $End_k(U)$

$$(a_{i_1},\dots,a_{i_k})^c\to tr(X_{i_1}\cdots X_{i_k})$$

and the product of cyclic words goes to the product of traces. The commutator $I_\Lambda=[\Lambda,\cdot]$, for $\Lambda\in End(U)^{odd}$, is an odd differentiation of $End(U)$. Notice that $I_\Lambda^2\neq 0$ for generic Λ. For such Λ there always exists an operator I_Λ^{-1} of regularized inverse: $[I_\Lambda,I_\Lambda^{-1}]=1$, preserving the supertrace functional. A choice of nilpotent I_Λ^{-1} is in one-to-one correspondence with I_Λ^2-invariant Lagrangian subspace in $\Pi\,End(U)$, corresponding to $L=\{x\in M\mid I_\Lambda^{-1}(x)=0\}$. Let $\Lambda=\left(\begin{smallmatrix}0 & Id\\ \Lambda_{01} & 0\end{smallmatrix}\right)$, $\Lambda_{01}=diag(\lambda_1,\dots,\lambda_d)$, I take $I_\Lambda^{-1}\left(\begin{smallmatrix}X_{00} & X_{10}\\ X_{01} & X_{11}\end{smallmatrix}\right)=\left(\begin{smallmatrix}M_\lambda X_{01} & M_\lambda(X_{00}+X_{11})\\ 0 & -M_\lambda X_{01}\end{smallmatrix}\right)$ where $M_\lambda E_i^j=(\lambda_i+\lambda_j)^{-1}E_i^j$.

Theorem 1. *By the standard Feynman rules, the asymptotic expansion, at $\Lambda^{-1}\to 0$, is given by the following sum over oriented stable ribbon graphs*:

$$\log\frac{\int_L \exp\frac{1}{\hbar}(-\frac{1}{2}tr\circ B^{-1}([\Lambda,X],X)+S_{gl}(X))\,\mathrm{d}X}{\int_L \exp\frac{1}{\hbar}(-\frac{1}{2}tr\circ B^{-1}([\Lambda,X],X))\,\mathrm{d}X}=const\sum_G \hbar^{-\chi_G}c_S(G)c_\Lambda(G),$$

where χ_G is the Euler characteristic of the corresponding surface, $c_S(G)$ is my partition function associated with the solution S, $c_\Lambda(G)$ is the partition function associated with $(End(U),tr,I_\Lambda)$ and constructed using the propagator $tr(I_\Lambda^{-1}\cdot,\cdot)$. This partition function $c_\Lambda(G)$ defines the cohomology class in $H^*(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu,\mathcal{L})$.

This follows from the standard rules [6] of the Feynman diagrams calculus (compare with the formula (0.1) from [5]). In particular the combinatorics of the terms in $S_{gl}(X)$ matches the data associated with vertices in the complex $\mathcal{F}\widetilde{\mathbb{S}}$, i.e. the symmetric product of cyclic permutations and the integer number. The construction of $c_\Lambda(G)$, in the odd and even cases, is studied in more details in [3], see also [8] for some motivation, and [2] for further developments.

2.2. Even inner product

Let now S denotes a solution to quantum master equation in the Batalin–Vilkovisky algebra (1), with $S_{0,1,2}=S_{0,1,1}=0$. In this case my basic matrix algebra is the general *queer* superalgebra $q(U)$ [4] with its *odd* trace. The associative superalgebra $q(U)$ is the subalgebra of $End(U\oplus\Pi U)$,

$$q(U)=\big\{X\in End(U\oplus\Pi U)\;\big|\;[X,\pi]=0\big\},$$

where U is a purely even vector space and $\pi:U\rightleftarrows\Pi U$, $\pi^2=1$ is the odd operator changing the parity. As a vector space $q(U)=End(U)\oplus\Pi\,End(U)$. The odd trace on $q(U)$ is defined as $otr(X)=\frac{1}{2}tr(\pi X)$. Let us extend S to the function S_q on $M=Hom(\Pi V,q(U))$, so that each cyclically symmetric tensor in $\Pi(\Pi V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$ goes to the odd trace of the product of the corresponding elements from $q(U)$

$$(a_{i_1},\dots,a_{i_j})^c\to otr(X_{i_1}\cdots X_{i_j})$$

and the product of cyclic words goes to the product of the odd traces. Let us denote by $otr \circ B^{-1}$ the odd extension to M, which is defined using the *odd* pairing $otr(XX')$ on $q(U)$, of the even symmetric inner product on $Hom(V,k)$. Let $\Lambda = \Pi\, diag(\lambda_1, \ldots, \lambda_N)$, $\Lambda \in \Pi\, End(U)$ be the odd element from $q(U)$. The commutator $I_\Lambda = [\Lambda, \cdot]$ is an odd differentiation of $q(U)$, and for generic λ_i, $I_\Lambda^2 \neq 0$, and I_Λ is *invertible* outside of the even diagonal. Define the regularized inverse I_Λ^{-1}, $(I_\Lambda^{-1})E_i^j = (\lambda_i + \lambda_j)^{-1}\Pi E_i^j$, $(I_\Lambda^{-1})\Pi E_i^j = 0$, $[I_\Lambda, I_\Lambda^{-1}] = 1$, and let $L = \{x \in M \mid I_\Lambda^{-1}(x) = 0\}$.

Theorem 2. *By the standard Feynman rules, the asymptotic expansion, at $\Lambda^{-1} \to 0$, is given by the following sum over oriented stable ribbon graphs*:

$$\log \frac{\int_L \exp \frac{1}{\hbar}(-\frac{1}{2}\, otr \circ B^{-1}([\Lambda, X], X) + S_q(X))\, dX}{\int_L \exp(-\frac{1}{2\hbar}\, otr \circ B^{-1}([\Lambda, X], X))\, dX} = const \sum_G \hbar^{-\chi_G} c_S(G) c_\Lambda(G),$$

where $c_S(G)$ is the partition function from [1] *and $c_\Lambda(G)$ is the partition function, associated with $(q(U), otr, I_\Lambda)$, and constructed using the propagator $otr^{dual}(I_\Lambda^{-1}\cdot, \cdot)$. This partition function $c_\Lambda(G)$ defines the cohomology class in $H^*(\overline{\mathcal{M}}'_{\gamma,\nu}/\mathbb{S}_\nu)$, it is the generating function for the products of tautological classes.*

Theorem 3. *Let us consider the case of the one-dimensional vector space V with even symmetric inner product. The solutions in this case are arbitrary linear combination of cyclic words $X^3, X^5, \ldots, X^{2n+1}$. My matrix integral in this case is a supersymmetric extension of the Kontsevich matrix integral from* [7].

References

[1] S. Barannikov, Modular operads and non-commutative Batalin–Vilkovisky geometry, Internat. Math. Res. Notices 2007 (2007), Article ID: rnm075, Preprint MPIM (Bonn) 2006-48.
[2] S. Barannikov, Supersymmetric matrix integrals and σ-model, Preprint hal-00443592, 2009.
[3] S. Barannikov, Supersymmetry and cohomology of graph complexes, Preprint hal-00429963, 2009.
[4] J.N. Bernstein, D.A. Leites, The superalgebra $Q(n)$, the odd trace, and the odd determinant, Dokl. Bolg. Akad. Nauk 35 (3) (1982) 285–286.
[5] E. Getzler, M. Kapranov, Modular operads, Compositio Math. 110 (1) (1998) 65–126.
[6] P. Deligne, et al. (Eds.), Quantum Fields and Strings: A Course for Mathematicians, vols. 1, 2, AMS, Providence, RI, 1999.
[7] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys. 147 (1) (1992) 1–23.
[8] M. Kontsevich, Feynman diagrams and low-dimensional topology, in: Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 97–121.