C. R. Acad. Sci. Paris, Ser. I 348 (2010) 407–410

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Harmonic Analysis

A characterization of Fourier transform by Poisson summation formula

Une charactérisation de la transformation de Fourier par la formule sommatoire de Poisson

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ARTICLE INFO

Article history:
Received 15 November 2009
Accepted after revision 21 January 2010

Presented by Jean Bourgain

ABSTRACT

We show that, under certain conditions, the Fourier transform is completely characterized by Poisson's summation formula. Also, we propose a generalized transform which is derived from a Poisson-type summation formula, that we call a Fourier–Poisson transform.

RÉSUMÉ

Nous montrons que, sous certaines conditions, la transformation de Fourier est complétement characterisée par la formule sommatoire de Poisson. Nous proposons aussi une transformation généralisée qui est dérivée d'une formule de sommation de type Poisson; nous l'appelons la transformation de Fourier–Poisson.

1. Introduction

The classical summation formula of Poisson states that, for a well-behaved function $f:\mathbf{R}\to\mathbf{R}$ and its (suitably scaled) Fourier Transform $\hat{f}$ we have the relation

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

Fix $x>0$, and replace $f(t)$ with $\frac{1}{x}f(t/x)$. Also assume f is even and $f(0)=0$. Then

$$\sum_{n=1}^{\infty} \hat{f}(nx) = \frac{1}{x}\sum_{n=1}^{\infty} f(n/x) \tag{1}$$

It would be interesting to understand to which extent this summation formula, which involves sums over lattices in $\mathbf{R}$, determines the Fourier transform of a function.

In this article we prove that for a certain class of functions, the (scaled) Poisson summation formula completely determines its Fourier transform.

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[1] Supported in part by the Israel Science Foundation grant No. 387/09.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.01.023

Theorem 1. *Assume $f \in C^2$, and $f, f', f'' \in L_1[0,\infty)$. Assume also that $f(0) = \int_0^\infty f = 0$. Then there is a unique continuous function g on $x > 0$ satisfying $g(x) = O(x^{-1-\epsilon})$ for some $\epsilon > 0$ s.t. $\sum_{n=1}^\infty g(nx) = \frac{1}{x}\sum_{n=1}^\infty f(n/x)$. Moreover, $g = \hat{f}$.*

We would like to mention a different uniqueness result due to Cordoba [1]:

Theorem. *Suppose $\{x_k\}$ and $\{y_k\}$ are two discrete sets in $\mathbf{R}^n$, and for all f $\sum_k f(x_k) = \sum_k \hat{f}(y_k)$. Then x_k and y_k are dual lattices, i.e. there is a linear transformation A with $\det A = 1$ s.t. $\{x_k\} = A(\mathbf{Z}^n)$ and $\{y_k\} = (A^*)^{-1}(\mathbf{Z}^n)$.*

2. Some notation, and a non-formal treatment

Denote by δ_n, $n \geqslant 1$ the sequence given by $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. We define the convolution of sequences as $(a * b)_k = \sum_{mn=k} a_m b_n$, and the operator $T_{a_n} f(x) = \sum_{n=1}^\infty a_n f(nx)$.

It holds that $T_{b_n} T_{a_n} f = T_{a_n * b_n} f$ whenever the series in both sides are well defined and absolutely convergent.

Let a_n, b_n, $n \geqslant 1$ be two sequences, which satisfy $a * b = \delta$.

This is equivalent to saying that $L(s; a_n)L(s; b_n) = 1$ where $L(s; c_n) = \sum_{n=1}^\infty \frac{c_n}{n^s}$. For a given a_n, its convolutional inverse is uniquely defined via those formulas.

Then, the formal inverse transform to T_{a_n} is given simply by T_{b_n}. Note that the convolutional inverse of the sequence $a_n = 1$ is the Möbius function $\mu(n)$.

In terms of T, the Poisson summation formula for the Fourier transform can be written as following:

$$T_{a_n} \hat{f}(x) = \left(\frac{1}{x}\right)(T_{a_n} f)\left(\frac{1}{x}\right)$$

where $a_n = 1$ for all n. This suggests a formula for the Fourier transform:

$$\hat{f}(x) = T_{b_n}\left(\left(\frac{1}{x}\right)(T_{a_n} f)\left(\frac{1}{x}\right)\right)(x) \tag{2}$$

with $b_n = \mu(n)$ the Möbius function. These formulas hold for $x > 0$, which is our region of interest.

Later we will have a rigorous treatment of the convergence properties of this formal series. Meanwhile we would like to mention that Davenport in [2] established certain identities, such as

$$\sum_{n=1}^\infty \frac{\mu(n)}{n}\{nx\} = -\frac{1}{\pi}\sin(2\pi x)$$

which could be used to show that formula (2) actually produces the Fourier transform of a (zero-integral) step function.

We formally define the Fourier–Poisson transform associated with a_n (and its convolutional inverse b_n)

$$\mathcal{F}_{a_n} f(x) = T_{b_n}\left(\left(\frac{1}{x}\right) T_{a_n} f\left(\frac{1}{x}\right)\right)(x) \tag{3}$$

This is clearly an involution, and is the Fourier transform for $a_n = 1$ and the transform $f(x) \mapsto \frac{1}{x} f(\frac{1}{x})$ for $a_n = \delta_n$. Note that both are isometries of $L_2[0,\infty)$.

We can write the formal formula for $f(x) = x^{-s}$

$$\mathcal{F}_{a_n}\left(x^{-s}\right) = x^{s-1} L(s; a_n) L(1-s; b_n)$$

In particular, for $s = 1/2$, $x^{-1/2}$ is an eigenfunction with eigenvalue 1.

A multidimensional extension is straightforward: the sequence a_n is replaced by $a_{n_1,\dots,n_d}$, convolution is defined as $(a * b)_{k_1,\dots,k_d} = \sum_{m_j n_j = k_j} a_{n_1,\dots,n_d} b_{m_1,\dots,m_d}$. The tensorized sequence case $a_{n_1,\dots,n_d} = a_{n_1} \dots a_{n_d}$ is noteworthy. The multidimensional Fourier transform would be obtained by taking $a_{n_1,\dots,n_d} = 1$ and the convolutional inverse $b_{n_1,\dots,n_d} = \mu(n_1) \dots \mu(n_d)$.

3. Proof of Theorem 1

All functions are continuous, unless stated otherwise. The big-O notation $O(f(x))$ always refers to $x \to \infty$. We also employ Vinogradov's $f \ll g$ notation, which is equivalent to $f = O(g)$.

First we show the main results for the Fourier Transform, i.e. $a_n = 1$ and $b_n = \mu(n)$.

Lemma 1. *Suppose g is continuous in $(0,\infty)$ and satisfies $g(x) = O(x^{-1-\epsilon})$, $\epsilon > 0$. Then $T_{a_n} g = O(x^{-1-\epsilon})$ and $T_{b_n} g = O(x^{-1-\epsilon})$, and these are inverse transforms: $T_{a_n} T_{b_n} g = T_{b_n} T_{a_n} g = g$.*

Proof. First note that

$$T_{a_n} g(x) = \sum_{n=1}^{\infty} g(nx) \ll \frac{1}{x^{1+\epsilon}} \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} \ll \frac{1}{x^{1+\epsilon}}$$

Then we have absolute convergence in $T_{a_n} T_{b_n} g$, which allows changing order of summation for every fixed $x > 0$:

$$\sum_n \left|\mu(n)\right| \sum_m \left|f(mnx)\right| \ll x^{-1-\epsilon} \left(\sum_n n^{-1-\epsilon}\right)\left(\sum_m m^{-1-\epsilon}\right) \ll x^{-1-\epsilon}$$

Thus the formal inverse T_{b_n} is the actual inverse, and we are finished. □

Theorem 1. *Let $a_n = 1$, $f \in C^2$, and $f, f', f'' \in L_1[0,\infty)$. Assume also that $f(0) = \int_0^\infty f = 0$. Then there exists a unique continuous function g on $x > 0$ satisfying $g(x) = O(x^{-1-\epsilon})$ for some $\epsilon > 0$ s.t. $T_{a_n} g(x) = \frac{1}{x} T_{a_n} f(1/x)$.*

Proof. The Fourier transform of f, $\hat{f}$ is continuous and also $\hat{f} = O(x^{-2})$ due to smoothness of f. By the Poisson summation formula, and because $f(0) = \hat{f}(0) = 0$ we have $T_{a_n} \hat{f}(x) = \frac{1}{x} T_{a_n} f(\frac{1}{x})$. This proves existence. On the other hand, if g satisfies the conditions of the theorem, we have $T_{a_n} g = T_{a_n} \hat{f}$. By Lemma 1, $T_{a_n} \hat{f} = O(x^{-2})$. Again by that lemma, g must equal $\hat{f}$. □

4. A family of Fourier–Poisson transforms

Next we would like to state a similar result for a slightly different transform. It will justify the formally defined Fourier–Poisson operator that we introduced, for certain sequences and families of functions.

Fix the multiplicative function $a(x) = x^\lambda$, with $\lambda < -1/2$. The convolutional inverse of $a_n = a(n)$ is $b(n) = \mu(n) n^\lambda$.

Lemma 2. *If $g(x) = O(x^\mu)$, $\mu < -1/2$, then $T_{a_n} g = O(x^\mu)$ and $T_{b_n} g = O(x^\mu)$, and these are inverse transforms: $T_{a_n} T_{b_n} g = T_{b_n} T_{a_n} g = g$.*

Proof.

$$\left|T_{a_n} g(x)\right| = \left|\sum_n n^\lambda g(nx)\right| \ll x^\mu \sum_n n^{\lambda+\mu} = O\left(x^\mu\right)$$

similarly for T_{b_n}. That they are inverse for each other follows from absolute convergence of all series. □

Theorem 2. *Let $\lambda < -0.5$, $a_n = n^\lambda$, and assume that f satisfies $\int_0^\infty |f(x)| x^\lambda \, \mathrm{d}x < \infty$. Then*

(a) *The transform $\mathcal{F}_{a_n}$ defined in (3) is well-defined, and $\mathcal{F}_{a_n} f(x) = O(x^\lambda)$.*
(b) *If $g(x) = O(x^\lambda)$, and the Poisson summation formula holds, i.e. $T_{a_n} g(x) = \frac{1}{x} T_{a_n} f(\frac{1}{x})$, then $g = \mathcal{F}_{a_n} f$.*

Proof. (a) The series defining $\mathcal{F}_{a_n} f(x)$ converge absolutely:

$$\left|\sum_n \frac{b(n)}{nx} \sum_m a(m) f(m/nx)\right| \leqslant \sum_n a(nx) \left|b(n)\right| \sum_m a(m/nx) \left|f(m/nx)\right| 1/nx$$
$$\ll a(x) \sum_n n^{2\lambda} \int_0^\infty a(t) \left|f(t)\right| \mathrm{d}t \ll a(x) = x^\lambda$$

(b) By Lemma 2, we can apply T_{b_n} to both sides of the Poisson summation formula, and we are done. □

We would like to remark that for $a_n = n^\lambda$, $\lambda < -0.5$, $T_\lambda := T_{a_n}$ is actually a bounded operator on $L^2[0,\infty)$: Take a Schwartz function f. Then

$$\|T_\lambda f\|^2 = \sum_{m,n=1}^{\infty} m^\lambda n^\lambda \left\langle f(mx), f(nx)\right\rangle \leqslant \sum_{m,n=1}^{\infty} m^\lambda n^\lambda (mn)^{-1/2} \|f\|^2 \leqslant \zeta(1/2-\lambda)^2 \|f\|^2$$

thus $\|T_\lambda\| \leqslant \zeta(1/2-\lambda)$, and extends to a bounded operator on $L^2[0,\infty)$.

Acknowledgements

The author is grateful to Vitali Milman for suggesting this problem, and for the many fruitful discussions. Also, the author is grateful to Sasha Sodin for his ideas and remarks on the subject.

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