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Partial Differential Equations

Remarks on a polyharmonic eigenvalue problem

Remarques sur un problème poly-harmonique de valeurs propres

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ABSTRACT

This Note deals with a nonlinear eigenvalue problem involving the polyharmonic operator on a ball in $\mathbb{R}^n$. The main result of this Note establishes the existence of a continuous spectrum of eigenvalues such that the least eigenvalue is isolated.

RÉSUMÉ

On considère un problème non linéaire de valeurs propres associé à l'opérateur poly-harmonique sur une boule dans $\mathbb{R}^n$. Dans cette Note on montre l'existence d'un spectre continu de valeurs propres tel que la valeur propre principale est isolée.

Version française abrégée

Soit B une boule de rayon $R>0$ dans $\mathbb{R}^n$ et soit K un entier strictement positif. Dans cette Note on étudie le problème non linéaire de valeurs propres

$$\begin{cases} (-\Delta)^K u = \lambda f(x,u) & \text{dans } B, \\ u = Du = \cdots = D^{K-1}u = 0 & \text{sur } \partial B. \end{cases} \tag{1}$$

On suppose que λ est un paramètre positif et que la fonction f est définie par

$$f(x,t) = \begin{cases} t, & \text{si } t<0, \\ h(x,t), & \text{si } t \geqslant 0, \end{cases} \tag{2}$$

où $h : B \times \mathbb{R}_0^+ \to \mathbb{R}$ est une fonction de Carathéodory telle que si $H(x,t) := \int_0^t h(x,s)\,\mathrm{d}s$, alors les conditions suivantes soient satisfaites :

(H_1) *Il existe* $c \in (0,1)$ *tel que* $|h(x,t)| \leqslant ct$ *pour tout* $t \in \mathbb{R}$ *et p.p.* $x \in B$;
(H_2) *Il existe* $t_0 > 0$ *tel que* $H(x,t_0) > 0$ *pour p.p.* $x \in B$;
(H_3) $\lim_{t\to\infty} \frac{h(x,t)}{t} = 0$ *uniformément sur* $B \setminus \mathcal{O}$, *où* $\mu(\mathcal{O}) = 0$.

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On démontre que les valeurs de λ pour lesquelles le problème (1) admet une solution sont liées à la première valeur propre du problème *linéaire*

$$\begin{cases} (-\Delta)^K u = \lambda u & \text{in } B, \\ u = Du = \cdots = D^{K-1}u = 0 & \text{on } \partial B. \end{cases} \tag{3}$$

Le résultat principal de cette Note est le suivant.

Théorème 0.1. *Supposons que la fonction f est du type* (2) *et satisfait les hypothèses* (H_1)–(H_3). *Alors la première valeur propre λ_1 du problème* (3) *est une valeur propre isolée du problème* (1) *et, de plus, l'ensemble correspondant de fonctions propres est un cône. En même temps, aucun $\lambda \in (0, \lambda_1)$ n'est une valeur propre du problème* (1) *et il existe $\mu_1 > \lambda_1$ tel que chaque $\lambda \in (\mu_1, \infty)$ est une valeur propre du problème* (1).

Les étapes principales dans la démonstration de ce résultat sont les suivantes :

(i) si $\lambda > 0$ est une valeur propre associée au problème (1), alors $\lambda \geqslant \lambda_1$;
(ii) la première valeur propre λ_1 du problème linéaire (3) est aussi une valeur propre du problème non linéaire (1) et, de plus, l'ensemble associé de fonctions propres est un cône dans l'espace de Hilbert $H_0^K(B)$ muni du produit scalaire

$$\langle u, v\rangle_K = \begin{cases} \int_B (\Delta^m u)(\Delta^m v)\,\mathrm{d}x, & \text{si } K = 2m, \\ \int_B (D\Delta^m u)(D\Delta^m v)\,\mathrm{d}x, & \text{si } K = 2m+1\,; \end{cases}$$

(iii) λ_1 est isolée dans l'ensemble de valeurs propres du problème (1);
(iv) il existe $\lambda^* > 0$ tel que $\inf_{H_0^K(B)} I_\lambda(u) < 0$ pour tout $\lambda \geqslant \lambda^*$, où

$$I_\lambda(u) := \frac{1}{2}\|u\|_K^2 - \lambda \int_B H(x, u_+)\,\mathrm{d}x, \quad u \in H_0^K(B)$$

est l'énergie associée au problème (1).

1. Introduction

Let B be any ball of $\mathbb{R}^n$ centered at the origin and of fixed radius $R > 0$. Consider the linear eigenvalue problem

$$\begin{cases} (-\Delta)^K u = \lambda u & \text{in } B, \\ u = Du = \cdots = D^{K-1}u = 0 & \text{on } \partial B, \end{cases} \tag{4}$$

where K is a positive integer. Then the lowest eigenvalue λ_1 of problem (4) is *simple*, that is, the associated eigenfunctions are merely multiples of each other. Moreover they are radial, strictly monotone in $r = |x|$ and never change sign in B. We refer to Pucci and Serrin [3] for further properties of eigenvalues of polyharmonic operators.

In this paper we are concerned with the nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)^K u = \lambda f(x,u) & \text{in } B, \\ u = Du = \cdots = D^{K-1}u = 0 & \text{on } \partial B, \end{cases} \tag{5}$$

where λ is a positive parameter and the nonlinear function f is given by

$$f(x,t) = \begin{cases} t, & \text{if } t < 0, \\ h(x,t), & \text{if } t \geqslant 0, \end{cases} \tag{6}$$

where $h : B \times \mathbb{R}_0^+ \to \mathbb{R}$ is a Carathéodory function, $H(x,t) := \int_0^t h(x,s)\,\mathrm{d}s$, and the following conditions are fulfilled:

(H_1) *There exists $c \in (0,1)$ such that $|h(x,t)| \leqslant ct$ for all $t \in \mathbb{R}$ and a.a. $x \in B$;*
(H_2) *There exists $t_0 > 0$ such that $H(x,t_0) > 0$ for a.a. $x \in B$;*
(H_3) $\lim_{t\to\infty} \frac{h(x,t)}{t} = 0$ *uniformly in $B \setminus \mathcal{O}$, with $\mu(\mathcal{O}) = 0$.*

As already highlighted in [2], functions h verifying (H_1)–(H_3) are given in $B \times \mathbb{R}_0^+$, e.g., by $h(x,t) = \sin(ct)$, $h(x,t) = c\log(1+t)$, $h(x,t) = g(x)[t^{q(x)-1} - t^{p(x)-1}]$, where $c \in (0,1)$, p, $q : \overline{B} \to (1,2)$ continuous in $\overline{B}$, $\max_{\overline{B}} p(x) < \min_{\overline{B}} q(x)$, $g \in L^\infty(B)$, $\|g\|_\infty = c$. For the relevance of these examples in applications, as well as for a wide list of references, we refer to [2].

The main result of this Note is the following:

Theorem 1.1. *Suppose that f is of type* (6) *and that* (H_1)–(H_3) *are fulfilled. Then the first eigenvalue λ_1 of* (4) *is an isolated eigenvalue of problem* (5) *and the corresponding set of eigenfunctions is a cone. Moreover, any $\lambda \in (0, \lambda_1)$ is not an eigenvalue of* (5)*, while there exists $\mu_1 > \lambda_1$ such that any $\lambda \in (\mu_1, \infty)$ is an eigenvalue of* (5)*.*

2. Proof of Theorem 1.1

Consider the standard higher order Hilbertian Sobolev space $H_0^K = H_0^K(B)$, endowed with the scalar product

$$\langle u, v\rangle_K = \begin{cases} \int_B (\Delta^m u)(\Delta^m v)\,\mathrm{d}x, & \text{if } K = 2m,\\ \int_B (D\Delta^m u)(D\Delta^m v)\,\mathrm{d}x, & \text{if } K = 2m+1,\end{cases}$$

and denote by $\|\cdot\|_K$ the corresponding norm. As in [1, Section 3], the decomposition method of Moreau and the comparison principle of Boggio in H_0^K substitute the decomposition in the positive and negative part which is no longer admissible in the higher order Sobolev spaces when $K \geqslant 2$. Indeed, for any $u \in H_0^K$ there exists a unique couple $(u_1, u_2) \in \mathcal{K} \times \mathcal{K}'$ such that $u = u_1 + u_2$ and $\langle u_1, u_2\rangle_K = 0$, where $\mathcal{K}$ is the convex closed cone of positive functions

$$\mathcal{K} = \{v \in H_0^K\colon\ v(x) \geqslant 0 \text{ a.e. in } B\},$$

while $\mathcal{K}'$ is the dual cone of $\mathcal{K}$, that is

$$\mathcal{K}' = \{w \in H_0^K\colon\ \langle w, v\rangle_K \leqslant 0 \text{ for all } v \in \mathcal{K}\}.$$

By [1, Lemma 2] we know that $\mathcal{K}'$ is contained in the cone of negative functions, in other words $w(x) \leqslant 0$ a.e. in B if $w \in \mathcal{K}'$.

The number $\lambda > 0$ is an eigenvalue of problem (5), with f of the type (6), if there exists $u \in H_0^K \setminus \{0\}$ such that

$$\langle u, v\rangle_K = \lambda \int_B f(x, u) v\,\mathrm{d}x \tag{7}$$

for any $v \in H_0^K$.

Lemma 2.1. *If $\lambda > 0$ is an eigenvalue of* (5)*, then $\lambda \geqslant \lambda_1$.*

Proof. Assume that $\lambda > 0$ is and eigenvalue of (5), with corresponding eigenfunction $u \in H_0^K \setminus \{0\}$. Letting $v = u$ in (7), and putting $B_- = \{x \in B\colon\ u(x) \leqslant 0\}$ and $B_+ = \{x \in B\colon\ u(x) \geqslant 0\}$, we get by (H_1)

$$\|u\|_K^2 = \lambda\left[\int_{B_+} h(x, u)u\,\mathrm{d}x + \int_{B_-} u^2\,\mathrm{d}x\right] \leqslant \lambda\left[c\int_{B_+} u^2\,\mathrm{d}x + \int_{B_-} u^2\,\mathrm{d}x\right] \leqslant \lambda|u|_2^2,$$

being $c \in (0, 1)$. By the definition of λ_1

$$\lambda_1|u|_2^2 \leqslant \|u\|_K^2 \leqslant \lambda|u|_2^2.$$

Since $u \neq 0$, then the above inequality shows that $\lambda \geqslant \lambda_1$. □

Lemma 2.2. *The first eigenvalue λ_1 of* (4) *is also an eigenvalue of* (5) *and the set of the corresponding eigenfunctions is a cone of H_0^K.*

Proof. As already noted in the introduction the lowest eigenvalue λ_1 of (4) is simple, so that there exists a first eigenfunction $\varphi \in H_0^K \setminus \{0\}$, with $\varphi < 0$ in B. Hence φ is an eigenfunction also of (5), since clearly satisfies (7) with $\lambda = \lambda_1$, being $\langle \varphi, v\rangle_K = \lambda_1 \int_B \varphi v\,\mathrm{d}x = \lambda_1 \int_B f(x, \varphi) v\,\mathrm{d}x$ by (6). Moreover the set of the corresponding eigenfunctions lies in a cone of H_0^K. □

Lemma 2.3. *The first eigenvalue λ_1 of* (4) *is isolated in the set of eigenvalues of* (5)*.*

Proof. Let $\lambda > 0$ be an eigenvalue of (5) whose corresponding eigenfunction u has Moreau's decomposition with $u_1 \not\equiv 0$. Then, being $u_1 \in H_0^K$, we take $v = u_1$ in (7), and by the definition of λ_1 and (H_1) we get

$$\lambda_1|u_1|_2^2 \leqslant \|u_1\|_K^2 = \lambda\left[\int_{B_+} h(x, u)u_1\,\mathrm{d}x + \int_{B_-} uu_1\,\mathrm{d}x\right] \leqslant \lambda c|u_1|_2^2.$$

Hence $\lambda \geqslant \lambda_1/c > \lambda_1$, being $c \in (0, 1)$. In particular, any eigenfunction u corresponding to an eigenvalue $\lambda \in (0, \lambda_1/c)$ has decomposition $u = u_2$, so that u is also an eigenfunction of (4), since $u = u_2 \leqslant 0$ a.e. in B. It is known, as noted in the introduction, that $\lambda_1 < \lambda_2$, where λ_2 is the second eigenvalue of (4). Hence any $\lambda \in (\lambda_1, \delta)$, with $\delta = \min\{\lambda_1/c, \lambda_2\}$, cannot be eigenvalue of (4) and in turn is not an eigenvalue of (5), by the argument above. This completes the proof. □

As already noted, $\lambda > 0$ is an eigenvalue of the problem

$$\begin{cases} (-\Delta)^K u = \lambda h(x, u_+) & \text{in } B, \\ u = Du = \cdots = D^{K-1} u = 0 & \text{on } \partial B, \end{cases} \tag{8}$$

if there exists $u \in H_0^K \setminus \{0\}$ such that $\langle u, v\rangle_K = \lambda \int_B h(x, u_+) v \, dx$ for all $v \in H_0^K$, that is if and only if u is a nontrivial critical point of the C^1 functional $I_\lambda : H_0^K \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|_K^2 - \lambda \int_B H(x, u_+) \, dx.$$

If $\lambda > 0$ is an eigenvalue of (8), with corresponding eigenfunction $u = u_1 + u_2$, then taking as test function $v = u_2$ by (H_1) we get, being $\langle u_1, u_2\rangle_K = 0$ and $h(x, 0) = 0$ a.e. in B,

$$\|u_2\|_K^2 = \langle u, u_2\rangle_K = \lambda \int_B h(x, u_+) u_2 \, dx = \lambda \int_{B_+} h(x, u) u_2 \, dx \leqslant 0,$$

being $u_2 \leqslant 0$ a.e. in B, that is $u = u_1 \geqslant 0$ in B and $u \neq 0$. In particular, *any eigenvalue* λ *of* (8) *is also an eigenvalue of* (5). Assumption (H_3) implies that for every $\lambda > 0$ there exists $C_\lambda > 0$ such that $\lambda H(x, t) \leqslant C_\lambda + \lambda_1 t^2/4$ for a.a. $x \in B$ and all $t \in \mathbb{R}$, where λ_1 is the first eigenvalue of (4). Hence, by the definition of λ_1, we have that for all $u \in H_0^K$

$$I_\lambda(u) \geqslant \frac{1}{2}\|u\|_K^2 - \frac{\lambda_1}{4}|u|_2^2 - C_\lambda|B| \geqslant \frac{1}{4}\|u\|_K^2 - C_\lambda|B|,$$

in other words I_λ *is bounded from below, weakly lower semi-continuous and coercive on* H_0^K.

Lemma 2.4. *There exists* $\lambda^* > 0$ *such that* $\inf_{H_0^K} I_\lambda(u) < 0$ *for all* $\lambda \geqslant \lambda^*$.

Proof. By (H_2) there exists $t_0 > 0$ such that $H(x, t_0) > 0$ a.e. in B. Let $\Omega \subset B$ be a compact subset, sufficiently large, such that $|B \setminus \Omega| < \int_\Omega H(x, t_0) \, dx / ct_0^2$, where $c \in (0, 1)$ is given in (H_1). Take $u_0 \in C_0^\infty(B)$, with $u_0(x) = t_0$ if $x \in \Omega$ and $0 \leqslant u_0(x) \leqslant t_0$ if $x \in B \setminus \Omega$. Hence, by (H_1),

$$\int_B H\big(x, u_0(x)\big) \, dx \geqslant \int_\Omega H(x, t_0) \, dx - ct_0^2|B \setminus \Omega| > 0,$$

and so $I_\lambda(u_0) < 0$ for $\lambda > 0$ sufficiently large. The lemma follows at once. □

Now, we return to the proof of Theorem 1.1. Since I_λ is bounded from below, weakly lower semi-continuous and coercive on H_0^K, then Lemma 2.3 and [4, Theorem 1.2] show that I_λ has a negative global minimum for $\lambda > 0$ sufficiently large. This means that all such λ are eigenvalues of problem (8) and, consequently, of (5). This fact and Lemmas 2.1–2.3 complete the proof of Theorem 1.1.

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