



Mathematical Problems in Mechanics/Calculus of Variations

The optimal compliance problem for thin torsion rods: A 3D-1D analysis leading to Cheeger-type solutions

Optimisation de la structure d'une poutre fine en torsion et ensembles de Cheeger

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ABSTRACT

We consider the variational problem which consists in minimizing the compliance of a prescribed amount of isotropic elastic material placed into a given design region when it is subjected to a given load. We perform the asymptotics of this problem when the design region is a straight cylinder with infinitesimal cross section. The results presented in this Note concern the pure torsion regime and state the existence of optimal shapes for the limit problem. When the filling ratio tends in turn to zero, these optimal shapes concentrate on the boundary of the Cheeger set of the section of the design region.

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R É S U M É

On considère le problème d'optimisation suivant : une quantité fixée d'un matériau élastique isotrope donné doit être placée dans un cylindre droit de manière à maximiser sa résistance à un chargement donné tendant à provoquer un mouvement de torsion. Lorsque le rayon et le taux de remplissage du cylindre tendent tous deux vers zéro, on montre que la distribution optimale de matière se concentre dans chaque section sur le bord de l'ensemble de Cheeger.

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Version française abrégée

Nous considérons une suite de problèmes d'optimisation de forme où la région de design est un cylindre de la forme $Q_\delta = \delta D \times I$, où $\delta > 0$ est un petit paramètre, D est un ouvert borné et connexe de \mathbb{R}^2 et $I = [-1/2, 1/2]$. Un matériau élastique linéaire isotrope (caractérisé par son potentiel élastique de déformation $j(z) = (\lambda/2)(\text{tr}(z))^2 + \eta|z|^2$) doit être réparti de façon optimale dans un domaine $\Omega \subset Q_\delta$ pour résister à un chargement de la forme $G^\delta(x) := \delta^{-1}G(\delta^{-1}x', x_3) \in H^{-1}(Q_\delta; \mathbb{R}^3)$ où G est un élément de $H^{-1}(Q; \mathbb{R}^3)$ vérifiant la condition d'équilibre (1). La résistance de la structure est d'autant plus grande que la compliance, c'est à dire l'énergie élastique accumulée à l'équilibre, est faible. Cette compliance dont l'expression est décrite par (2) doit être minimisée parmi les domaines Ω contenus dans la région de design Q_δ et dont le volume est fixé. La contrainte de volume peut être traitée par l'introduction d'un multiplicateur de Lagrange k et, après mise à l'échelle, le problème s'écrit sous la forme (3). S'agissant d'une étude asymptotique de type 3D-1D, il est

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habituel d'effectuer un changement de variables afin de se ramener à un espace fonctionnel fixe. On définit l'opérateur e^δ et la compliance C^δ par

$$(e^\delta(U))_{\alpha\beta} := \delta^{-2}e_{\alpha\beta}(U), \quad (e^\delta(U))_{\alpha 3} := \delta^{-1}e_{\alpha 3}(U), \quad (e^\delta(U))_{33} := e_{33}(U), \quad \alpha, \beta \in \{1, 2\},$$

$$C_{j,G}^\delta(\omega) := \sup \left\{ \langle G, \delta^{-1}u + v + \delta w \rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u + \delta v + \delta^2 w)) \, dx : (u, v, w) \in BN(Q) \times T(Q) \times R_2^\perp(Q) \right\},$$

les espaces $BN(Q)$, $T(Q)$, $R_2^\perp(Q)$ étant introduits en Section 2. Le problème (3) devient alors

$$\phi_{j,G}^\delta(k) = \inf \{ C_{j,G}^\delta(\omega) + k|\omega| : \omega \subseteq Q \}.$$

Nous étudions d'abord le comportement asymptotique de ce problème lorsque, à k fixé, le paramètre δ tend vers zéro. Cela correspond à une réduction de dimension 3D-1D conduisant à un modèle unidimensionnel de poutre. Ce processus n'est pas standard car le cylindre contient des zones vides et aucune contrainte n'est faite sur la topologie de l'ensemble ω occupé par le matériau. On ne peut donc appliquer les résultats de la très importante littérature consacrée aux problèmes de réduction de dimension en élasticité, littérature que nous ne décrivons pas ici et pour laquelle nous renvoyons à [8]. Une étude similaire à celle traitée ici a été effectuée dans le cadre de la réduction 3D-2D (optimisation de la compliance de plaques) dans [1–4]. Les résultats annoncés dans cette note sont cependant très différents et offrent des géométries optimales plus riches. On se restreint à des chargements G satisfaisant l'hypothèse (4) de manière à ne solliciter la poutre qu'en torsion et l'hypothèse (5) afin d'utiliser les résultats de relaxation de [4]. On obtient le Théorème 3.1 qui donne le problème limite sous la forme (6)–(8) définissant la compliance optimale limite $\phi(k)$. Dans (8), $\bar{G}(x_3)$ désigne le moment des forces sur la section correspondante. Une formulation équivalente pour $\phi(k)$ est donnée par le Théorème 3.2.

Dans une seconde étape, nous étudions le comportement asymptotique de $\phi(k)$ lorsque k tend vers l'infini (ce qui correspond à un taux de remplissage tendant vers zéro). Le nouveau problème limite est donné par le Théorème 4.1, dans lequel la compliance limite a été étendue à toute mesure positive μ en remplaçant simplement $\theta \, dx$ par $d\mu$ dans (8). La formulation duale de ce problème, donnée par le Théorème 4.2, montre que l'optimisation peut être effectuée section par section. On peut alors chercher les solutions sous la forme $\gamma(x_3) \, dx_3 \otimes \nu(dx')$ où $\gamma(x_3) = \frac{1}{4\sqrt{\eta}} \int_0^{x_3} \bar{G}(s) \, ds$ et ν est une mesure sur D solution du problème bidimensionnel

$$\min \left\{ \int |\nu| : \nu \in \mathcal{M}(D; \mathbb{R}^2), \operatorname{div} \nu = 0, \int_D (x_1 \, d\nu_2(x') - x_2 \, d\nu_1(x')) = -2 \right\}.$$

En introduisant un potentiel u tel que $D_1 u = \nu_2$ et $D_2 u = -\nu_1$, on reformule ce problème sous la forme (10). D étant supposé convexe, il existe une unique solution qui est la fonction caractéristique de l'ensemble de Cheeger C de D , c'est à dire le sous ensemble de D qui minimise le rapport périmètre/aire.

Ainsi la mesure ν optimale est donnée par $\nu = \frac{1}{|C|} \mathcal{H}^1 \llcorner \partial C$. Une étude numérique montre qu'en effet, lorsque k est assez grand, la structure optimale se concentre au voisinage du bord de C (cf. Fig. 1).

1. Setting of the problem

The aim of this Note is to announce some new results about the following shape optimization problem: minimize the compliance of a vanishing amount of elastic material, in pure torsion regime, and confined into an asymptotically thin cylinder. A precise formulation can be given as follows.

Let Q be a cylinder of the form $Q = \bar{D} \times I$, where D is an open bounded connected region of \mathbb{R}^2 and $I = [-1/2, 1/2]$. Let $G \in H^{-1}(Q; \mathbb{R}^3)$ be a balanced load:

$$G = (G_1, G_2, G_3), \quad \text{with } \langle G, U \rangle_{\mathbb{R}^3} = 0 \quad \text{whenever } e(U) := (\nabla U + (\nabla U)^T)/2 = 0. \tag{1}$$

When an isotropic elastic material, characterized by a strain potential of the form $j(z) = (\lambda/2)(\operatorname{tr}(z))^2 + \eta|z|^2$, is distributed into a domain $\Omega \subseteq Q$, its compliance $C_{j,G}(\Omega)$ is given by

$$C_{j,G}(\Omega) := \sup \left\{ \langle G, U \rangle_{\mathbb{R}^3} - \int_\Omega j(e(U)) \, dx : U \in H^1(Q; \mathbb{R}^3) \right\}. \tag{2}$$

Consider now the problem of minimizing $C_{j,G}(\Omega)$ (which means maximizing the resistance to G), when the volume of Ω (i.e. the total amount of material) is prescribed, and Ω varies among the subsets of the thin design region $Q_\delta := \delta \bar{D} \times I$.

If the volume constraint is enclosed in the cost through a Lagrange multiplier $k \in \mathbb{R}$, the problem reads:

$$\phi_{j,G}^\delta(k) := \inf \left\{ C_{j,G}(\Omega) + \frac{k}{\delta^2} |\Omega| : \Omega \subseteq Q_\delta \right\}. \tag{3}$$

The load G^δ appearing in the r.h.s. of (3) is a scaling of G chosen so that in the limit process the infimum will remain finite and will involve only pure torsion displacements: writing any $x \in \mathbb{R}^3$ as $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, it is defined by $G_\alpha^\delta(x) := \delta^{-1}G_\alpha(\delta^{-1}x', x_3)$, $G_3^\delta(x) := \delta^{-2}G_3(\delta^{-1}x', x_3)$.

Our goal is to determine the asymptotics of the shape optimization problem (3) in a double limit process.

In a first step, for any fixed k , we investigate the limit $\phi(k)$ of $\phi_{j,G}^\delta(k)$ as $\delta \rightarrow 0^+$. Clearly, this is a 3D-1D dimension reduction, leading to an optimal thin rod model. We emphasize that, due to the presence of voids in the cylinder and to the lack of topological information on the set ω occupied by the material, such a dimension reduction is not covered by the extensive literature on 3D-1D analysis (among which we limit ourselves to quote [8] and references therein).

In a second step, we study the limit behaviour of $\phi(k)$ as $k \rightarrow +\infty$. As explained in [4, Section 6], this means that the “filling ratio” $|\Omega|/|Q|$ becomes infinitesimal.

The analogous of problem (3) in the 3D-2D case (namely when the design region is of the form $\bar{D} \times \delta I$), has been studied in some of our recent papers, see [1–4]. In this respect, we stress that the 3D-1D case is *not* a purely technical variant. In spite, it seems to be much more rich from a mathematical point of view: since now the infinitesimal sections of the design are no longer one but two-dimensional, optimal material distributions may take a wider variety of shapes. And, as shown in next sections, this opens the way to more interesting and surprising geometric effects.

2. Scaling

Let us introduce some preliminary notation. We let the Greek indices α and β run from 1 to 2, the index i run from 1 to 3, and we adopt the usual convention for the sums over repeated indices. For $x' = (x_1, x_2)$, we set $x'_\perp := (-x_2, x_1)$. We assume that the origin and the axes are chosen so that $\int_D x_\alpha dx' = 0$. We indicate by a dot the derivation of functions depending only on x_3 . We set $H_m^1(I)$ the space of functions in $H^1(I)$ having zero integral mean on I , and $H_m^2(I)$ the space of functions which belong to $H_m^1(I)$ together with their first derivative. Similarly, we denote by $H_m^1(D)$ the space of functions in $H^1(D)$ having zero integral mean on D . We introduce the following subspaces of $H^1(Q; \mathbb{R}^3)$ (cf. [7]), which correspond respectively to so-called Bernoulli–Navier fields, to twist displacements, and to the orthogonal of 2D-rigid motions:

$$\begin{aligned} BN(Q) &:= \{u: \exists (\zeta_\alpha, \zeta_3) \in (H_m^2(I))^2 \times H_m^1(I), u_\alpha(x) = \zeta_\alpha(x_3), u_3(x) = \zeta_3(x_3) - x_\alpha \dot{\zeta}_\alpha(x_3)\}, \\ T(Q) &:= \{v: \exists c \in H_m^1(I), v_\alpha(x) = c(x_3)x'_\perp, v_3 \in L^2(I, H_m^1(D))\}, \\ R_2^\perp(Q) &:= \{w: w_\alpha \in L^2(I, H_m^1(D)), -x_1 w_2 + x_2 w_1 \in H_m^1(D) \text{ for a.e. } x_3 \in I, w_3 = 0\}. \end{aligned}$$

Now, given any $U \in H^1(Q; \mathbb{R}^3)$, up to subtracting a 3D-rigid motion (which is not restrictive in problem (2)), it may be decomposed as $U = u + v + w$, where $u \in BN(Q)$, $v \in T(Q)$ and $w \in R_2^\perp(Q)$ are given by

$$\begin{aligned} u_\alpha(x) &= \zeta_\alpha(x_3), \quad u_3(x) = \zeta_3(x_3) - x_\alpha \dot{\zeta}_\alpha(x_3), \quad \text{with } \zeta_i(x_3) := \frac{1}{|D|} \int_D U_i(x', x_3) dx', \\ v_\alpha(x) &= c(x_3)x'_\perp, \quad \text{with } c(x_3) := \frac{1}{\int_D |x'|^2 dx'} \int_D (-x_2 U_1 + x_1 U_2) dx', \quad v_3(x) := U_3(x) - u_3(x), \\ w_\alpha(x) &:= U_\alpha(x) - u_\alpha(x) - v_\alpha(x), \quad w_3 := 0. \end{aligned}$$

We are now in a position to proceed with the scaling of problem (3). Firstly, we write any admissible domain $\Omega \subseteq Q_\delta$ in (3) as $\Omega = \{(\delta x', x_3): (x', x_3) \in \omega\}$ (so that $\omega \subseteq Q$). Then, denoting by $\tilde{U} \in H^1(Q_\delta; \mathbb{R}^3)$ an admissible displacement in the definition of $\mathcal{C}_{j,G^\delta}(\Omega)$, we decompose it as $\tilde{U} = \tilde{u} + \tilde{v} + \tilde{w}$ according to the above described procedure, and we set $\tilde{u}(x) := (\delta^{-2}u_\alpha(\delta^{-1}x', x_3), \delta^{-1}u_3(\delta^{-1}x', x_3))$, $\tilde{v}(x) := (\delta^{-1}v_\alpha(\delta^{-1}x', x_3), v_3(\delta^{-1}x', x_3))$, and $\tilde{w}(x) := (w_\alpha(\delta^{-1}x', x_3), 0)$. With these definitions, thanks to the scaling chosen for the load, we have $\langle G^\delta, \tilde{U} \rangle_{\mathbb{R}^3} = \delta^{-1} \langle G, u \rangle_{\mathbb{R}^3} + \langle G, v \rangle_{\mathbb{R}^3} + \langle G, w \rangle_{\mathbb{R}^3}$. Moreover, via change of variables, we have $\int_{Q_\delta} j(e(\tilde{U})) dx = \int_Q j(e^\delta(u + \delta v + \delta^2 w)) dx$, where, for any $U \in H^1(Q; \mathbb{R}^3)$, $e^\delta(U)$ is the rescaled strain tensor defined by $(e^\delta(U))_{\alpha\beta} := \delta^{-2}e_{\alpha\beta}(U)$, $(e^\delta(U))_{\alpha 3} := \delta^{-1}e_{\alpha 3}(U)$ and $(e^\delta(U))_{33} := e_{33}(U)$. Thus, setting

$$\mathcal{C}_{j,G}^\delta(\omega) := \sup \left\{ \langle G, \delta^{-1}u + v + \delta w \rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u + \delta v + \delta^2 w)) dx: (u, v, w) \in BN(Q) \times T(Q) \times R_2^\perp(Q) \right\},$$

we end up with the rescaled version of problem (3) on Q : $\phi_{j,G}^\delta(k) = \inf\{\mathcal{C}_{j,G}^\delta(\omega) + k|\omega|: \omega \subseteq Q\}$.

3. Limit compliance problem in pure torsion regime

As we focus on pure torsion regime, we consider only actions G which satisfy

$$G_3 = 0 \quad \text{and} \quad \llbracket G_1 \rrbracket = \llbracket G_2 \rrbracket = 0. \quad (4)$$

Here and below, for a given real measure ν on Q , $\llbracket \nu \rrbracket$ denotes real measure on I defined by the identity $\langle \llbracket \nu \rrbracket, \varphi \rangle_{\mathbb{R}} := \langle \nu, \varphi \rangle_{\mathbb{R}^3}$ for all $\varphi \in C^\infty(\mathbb{R}; \mathbb{R})$. Assumption (4) implies that, for any $u \in BN(Q)$, $\langle G, u \rangle_{\mathbb{R}^3} = 0$, and that, for any $v \in T(Q)$, $\langle G, v \rangle_{\mathbb{R}^3} = \langle \bar{G}, c \rangle_{\mathbb{R}}$, being c associated with v as in the definition of $T(Q)$ and \bar{G} the 1D-average momentum defined by $\bar{G} := \llbracket x_1 G_2 - x_2 G_1 \rrbracket$.

Further, we ask the following regularity condition on the topological support of G , which is needed for the validity of Proposition 2.8 in [2] and which is satisfied for instance when $\text{spt}(G)$ is a 2-rectifiable set:

$$K_\delta := \{x \in Q : \text{dist}(x, \text{spt}(G)) < \delta\} \text{ has vanishing Lebesgue measure as } \delta \rightarrow 0. \tag{5}$$

For all $z \in \mathbb{R}^3$ we set $\bar{j}(z) := \inf\{j(\sum_i z_i(e_i \otimes e_3 + e_3 \otimes e_i) + \sum_{\alpha, \beta} \xi_{\alpha\beta}(e_\alpha \otimes e_\beta + e_\beta \otimes e_\alpha)) : \xi_{\alpha\beta} \in \mathbb{R}\}$. Recalling that $j(z) = (\lambda/2)|\text{tr}(z)|^2 + \eta|z|^2$, some explicit computations give $\bar{j}(z) = 2\eta \sum_\alpha |z_\alpha|^2 + (Y/2)|z_3|^2$, where $Y = \eta \frac{3\lambda+2\eta}{\lambda+\eta}$ is the Young modulus, written in terms of the Lamé coefficients λ, η .

Theorem 3.1. *Let $G \in H^{-1}(Q; \mathbb{R}^3)$ satisfy assumptions (1), (4), and (5), and we let δ tend to zero. Then, for every fixed $k \in \mathbb{R}$, the sequence $\phi_{j,G}^\delta(k)$ converges to the limit $\phi(k)$ defined by*

$$\phi(k) := \inf \left\{ C^{\text{lim}}(\theta) + k \int_Q \theta \, dx : \theta \in L^\infty(Q; [0, 1]) \right\}, \tag{6}$$

where

$$C^{\text{lim}}(\theta) := \sup \left\{ \langle G, v \rangle_{\mathbb{R}^3} - \int_Q \bar{j}(e_{13}(v), e_{23}(v), e_{33}(u)) \theta \, dx : v \in T(Q), u \in BN(Q) \right\} \tag{7}$$

$$= \sup \left\{ \langle \bar{G}, c \rangle_{\mathbb{R}} - 2\eta \int_I |\dot{c}x'_\perp + \nabla_{x'} v_3|^2 \theta \, dx : c \in H_m^1(I), v_3 \in L^2(I; H_m^1(D)) \right\}. \tag{8}$$

Moreover, if $\omega^\delta \subseteq Q$ is a sequence of domains such that $\phi_{j,G}^\delta(k) = C_{j,G}^\delta(\omega^\delta) + k|\omega^\delta| + o(1)$, up to subsequences $\mathbb{1}_{\omega^\delta}$ converges weakly* in $L^\infty(Q; [0, 1])$ to a solution $\bar{\theta}$ of problem (6).

Theorem 3.2. *For every $k \in \mathbb{R}$, it holds*

$$\begin{aligned} \phi(k) &= \sup \left\{ \langle G, v \rangle_{\mathbb{R}^3} - \int_Q [\bar{j}(e_{13}(v), e_{23}(v), e_{33}(u)) - k]_+ \, dx : v \in T(Q), u \in BN(Q) \right\} \\ &= \sup \left\{ \langle \bar{G}, c \rangle_{\mathbb{R}} - \int_I \int_D [2\eta |\dot{c}x'_\perp + \nabla_{x'} v_3|^2 - k]_+ \, dx' \, dx_3 : c \in H_m^1(I), v_3 \in L^2(I; H_m^1(D)) \right\}. \end{aligned}$$

Remark 1. The search of a solution for $\phi(k)$ of the type $\theta_k = \mathbb{1}_{\omega_k}$ can be performed by comparing the expressions for $\phi(k)$ given in Theorems 3.1 and 3.2, and by writing the optimality conditions.

4. The vanishing filling ratio limit

The limit compliance $C^{\text{lim}}(\theta)$ given by Theorem 3.1 can be extended in a natural way to the class $\mathcal{M}^+(Q)$ of positive measures μ on \mathbb{R}^3 compactly supported in Q : simply define $C^{\text{lim}}(\mu)$ by the same formula as (7), with the measure $\theta \, dx$ replaced by $d\mu$. Then, the r.h.s. of (6) can be rewritten as the infimum of $C^{\text{lim}}(\mu) + k \int d\mu$, over the class of competitors μ of the form $\theta \, dx$, with $\theta \in L^\infty(Q; [0, 1])$. Multiplying μ by $\sqrt{2k}$ (for $k > 0$) leads to

$$\phi(k) = \sqrt{2k} \inf \left\{ C^{\text{lim}}(\mu) + \frac{1}{2} \int d\mu : \mu = \theta \, dx, \theta \in L^\infty(Q; [0, \sqrt{2k}]) \right\}.$$

Theorem 4.1. *As $k \rightarrow +\infty$, $\frac{\phi(k)}{\sqrt{2k}}$ converges decreasingly to*

$$\bar{m} := \inf \left\{ C^{\text{lim}}(\mu) + \frac{1}{2} \int d\mu : \mu \in \mathcal{M}^+(Q) \right\}. \tag{9}$$

Moreover, if ω_k is an optimal set for $\phi(k)$, as $k \rightarrow +\infty$, up to subsequences $\mathbb{1}_{\omega_k}$ converges weakly* in $L^\infty(Q; [0, 1])$ to a solution $\bar{\mu}$ of problem (9).

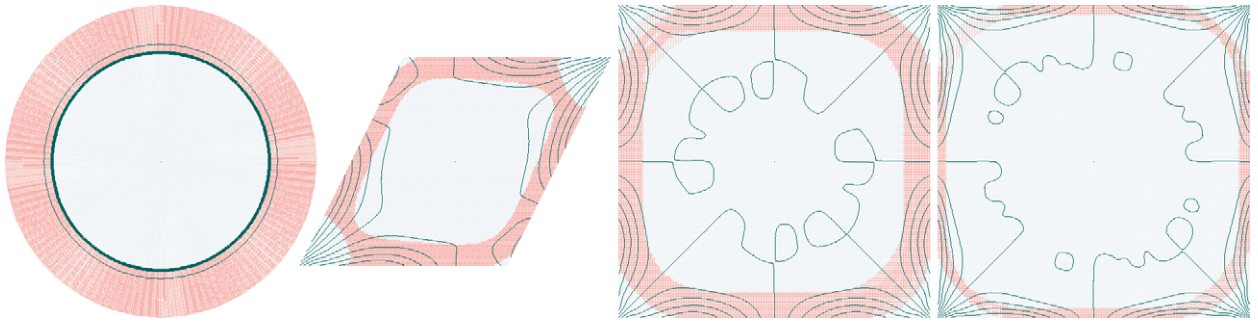


Fig. 1. The different design regions.

Theorem 4.2. *The infimum in (9) can be recast through the following problem in primal or dual form:*

$$\begin{aligned} \bar{m} &= \sup \left\{ \langle \bar{G}, c \rangle_{\mathbb{R}}, c \in H_m^1(I): \exists v_3 \in L^2(I; H_m^1(D)), |\dot{c}x_{\perp} + \nabla_{x'} v_3| \leq \frac{1}{2\sqrt{\eta}} \text{ a.e. on } Q \right\} \\ &= \min \left\{ \int |\sigma|: \sigma \in \mathcal{M}(Q; \mathbb{R}^2), \operatorname{div}_{x'} \sigma = 0, \llbracket x_1 \sigma_2 - x_2 \sigma_1 \rrbracket = -\frac{1}{2\sqrt{\eta}} \int_0^{x_3} \bar{G}(s) \, ds \right\}. \end{aligned}$$

Moreover, if $\bar{\sigma}$ is optimal for the above dual problem, then $\sum_{\alpha} \bar{\sigma}_{\alpha} (e_{\alpha} \otimes e_3 + e_3 \otimes e_{\alpha})$ is the limit stress tensor, and $\bar{\mu} := |\bar{\sigma}|$ is optimal for problem (9).

Clearly the minimization above for σ can be handled section by section, hence the solutions are of the form $\bar{\sigma} = \gamma(x_3) dx_3 \otimes \bar{v}(x')$, with $\gamma(x_3) := \frac{1}{4\sqrt{\eta}} \int_0^{x_3} \bar{G}(s) \, ds$, and \bar{v} solution to the following 2D problem: $\min\{ \int |\nu|: \nu \in \mathcal{M}(D; \mathbb{R}^2), \operatorname{div} \nu = 0, \int_D (x_1 \, dv_2(x') - x_2 \, dv_1(x')) = -2 \}$. Though less restrictive assumptions can be considered, here we assume for simplicity that D is convex. By writing any admissible ν as $(-D_2 u, D_1 u)$, for some u in $BV_0(D) := \{u \in BV(\mathbb{R}^2): u = 0 \text{ on } \mathbb{R}^2 \setminus \bar{D}\}$, we arrive at the problem

$$\min \left\{ \int |Du|: u \in BV_0(D), \int_D u = 1 \right\}. \tag{10}$$

By the convexity of D , the unique solution to (10) is given by $\bar{u} = |C|^{-1} \mathbb{1}_C$, where C is the so-called Cheeger set of D (see [5,6]). Correspondingly, the solution $\bar{\mu}$ to (9) is given by $\bar{\mu} = |\gamma(x_3)| dx_3 \otimes \frac{1}{|C|} \mathcal{H}^1 \llcorner \partial C$.

The parts of Fig. 1, corresponding to different design regions, have been obtained by numerical computations. The optimal shape is represented by the grey region. The last two pictures, corresponding to decreasing filling ratios, show the concentration occurring around the boundary of the Cheeger set.

References

[1] G. Bouchitté, I. Fragalà, Optimality conditions for mass design problems and applications to thin plates, Arch. Ration. Mech. Anal. 184 (2007) 257–284.
 [2] G. Bouchitté, I. Fragalà, Optimal design of thin plates by a dimension reduction for linear constrained problems, SIAM J. Control Optim. 46 (2007) 1664–1682.
 [3] G. Bouchitté, I. Fragalà, P. Seppecher, 3D-2D analysis for the optimal elastic compliance problem, C. R. Acad. Sci. Paris, Ser. I 345 (2007) 713–718.
 [4] G. Bouchitté, I. Fragalà, P. Seppecher, Structural optimization of thin plates: The three dimensional approach, preprint, 2009.
 [5] G. Carlier, M. Comte, On a weighted total variation minimization problem, J. Funct. Anal. 250 (2007) 214–226.
 [6] B. Kawohl, T. Lachand Robert, Characterization of Cheeger sets for convex subsets of the plane, Pacific J. Math. 225 (2006) 103–118.
 [7] F. Murat, A. Sili, Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces, C. R. Acad. Sci. Paris, Ser. I 328 (1999) 179–184.
 [8] L. Trabucho, J.M. Viaño, Mathematical Modelling of Rods, Handb. Numer. Anal., vol. IV, North-Holland, 1996, pp. 487–974.