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Number Theory

On certain exponential sums related to GL(3) cusp forms

Sur les sommes exponentielles associées aux GL(3)*-formes cuspidales*

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A R T I C L E   I N F O

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A B S T R A C T

We will prove a sharp estimate for certain exponential sums of Fourier coefficients of GL(3) cusp forms.

R É S U M É

On montre une estimation precise pour quelques sommes exponentielles des coefficients de Fourier des GL(3)-formes cuspidales.

1. Introduction

Let f be a GL(3) Maass form of type $\nu = (\nu_1, \nu_2)$ and a Hecke eigenform. It can be written in the form

$$f(z) = \sum_{\gamma \in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1, m_2)}{m_1 |m_2|} W_J \left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,1} \right),$$

where $U_2(\mathbb{Z})$ is the group of 2×2 upper triangular matrices with integer entries and ones on the diagonal, $W_J(z, \nu, \psi_{1,1})$ is the Jacquet–Whittaker function and $M = \operatorname{diag}(m_1|m_2|, m_1, 1)$. Assume that the form is normalized in such a way that the first Fourier coefficient is one. We are going to prove the following result:

Theorem 1.1. *Let $M^{2/3+\varepsilon} \ll \Delta \ll M$, let d be a fixed positive integer, and let $w(x)$ be a smooth weight function on the interval $[M, M+\Delta]$, with $w(x)^{(j)} \ll \Delta^{-j}$, for $0 \leqslant j \leqslant J$, for a sufficiently large J. Now*

$$\sum_{M \leqslant n \leqslant M+\Delta} A(1,n) w(n) e\left(\frac{d^{1/3} n}{M^{2/3}}\right) = -\frac{A(d,1)i}{\sqrt{3}\pi d^{-1/3}} \int_M^{M+\Delta} w(x) e\left(\frac{d^{1/3}}{M^{2/3}} x - 3x^{1/3} d^{1/3}\right) (dx)^{-1/3}\, \mathrm{d}x + O\left(\Delta M^{-2/3}\right),$$

where the constant implied by the O-notation depends on d.

As the integral on the right side of the equation is stationary when Δ is sufficiently small, we obtain the following corollary:

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Corollary 1.2. *Let $M^{2/3+\varepsilon} \ll \Delta \leqslant \frac{\sqrt{3}}{2} d^{-1/6} M^{5/6}$. Then*

$$\sum_{M \leqslant n \leqslant M+\Delta} A(1,n) w(n) e\left(\frac{d^{1/3} n}{M^{2/3}}\right) \asymp M^{-1/3} \Delta$$

if $A(d,1) \neq 0$, where $f \asymp g$ is understood to mean that both $f = O(g)$ and $g = O(f)$ hold. In particular, as $A(1,1) = 1$, the following holds:

$$\sum_{M \leqslant n \leqslant M+\Delta} A(1,n) w(n) e\left(\frac{n}{M^{2/3}}\right) \asymp M^{-1/3} \Delta.$$

When $\frac{\sqrt{3}}{2} d^{-1/6} M^{5/6} \leqslant \Delta \ll M$, by using the second derivative test and partial integration, we have the estimate $\ll M^{1/2}$ for the sum in question. On the other hand, we may use the triangle inequality to see that no better general upper bound can be obtained for sums of the same length. While the lower bounds obtained here are better than trivial bounds, they are weaker than the squareroot cancellation.

A similar question has been earlier tackled in the GL(2) setting by Karppinen and the author [1] together with estimates for short exponential sums. Namely, we proved that

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n) w(n) e\left(\frac{n}{\sqrt{M}}\right) \asymp M^{-1/4} \Delta,$$

when $M^{1/2+\varepsilon} \ll \Delta \ll M^{3/4}$ and $w(x)$ is a smooth weight function. Further, we showed that

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n) e(\alpha n) \ll \Delta M^{-1/4}$$

for any $\alpha \in [0,1]$ and $M^{3/4-1/32+\varepsilon} \ll \Delta \ll M^{3/4}$ while also proving other non-trivial estimates for short sums. Already 1987 Jutila [3] had proved the estimate

$$\sum_{n \leqslant M} a(n) e(\alpha n) \ll M^{1/2},$$

which is sharp.

The question of this article is very different in the sense that we do not have sharp upper bounds for long sums. Miller's [6] bound $M^{3/4+\varepsilon}$ is the best we know. However, the expected bound is $M^{1/2+\varepsilon}$.

Also, it is possible to prove something similar to Theorem 1.1 for sums over values of $A(p,n)$, where p is a fixed prime. However, the situation becomes messier. In general, if m is a fixed composite number, it is possible to obtain similar results but the more factors m has, the more difficult the situation.

2. Lemmas and preliminaries

Let us first recall the definition of a Kloosterman sum:

$$S(a,b;c) = \sum_{d\bar{d} \equiv 1 \ (\mathrm{mod}\ c)} e\left(\frac{da + \bar{d}b}{c}\right).$$

The Voronoi-type summation formula in GL(3) is the following [7,2]:

Lemma 2.1. *Let $\psi(x) \in C_c^\infty(0,\infty)$. Let $A(m,n)$ denote the (m,n)th Fourier coefficient of a Maass form for $SL(3,\mathbb{Z})$. Let $d, \bar{d}, c \in \mathbb{Z}$ with $c \neq 0$, $(d,c) = 1$, and $d\bar{d} \equiv 1 \ (\mathrm{mod}\ c)$. Then we have*

$$\sum_{n>0} A(m,n) e\left(\frac{n\bar{d}}{c}\right) \psi(n) = \frac{c\pi^{-5/2}}{4i} \sum_{n_1 \mid cm} \sum_{n_2>0} \frac{A(n_2,n_1)}{n_1 n_2} S\left(md, n_2; mcn_1^{-1}\right) \Psi_{0,1}^0\left(\frac{n_2 n_1^2}{c^3 m}\right)$$
$$+ \frac{c\pi^{-5/2}}{4i} \sum_{n_1 \mid cm} \sum_{n_2>0} \frac{A(n_2,n_1)}{n_1 n_2} S\left(md, -n_2; mcn_1^{-1}\right) \Psi_{0,1}^1\left(\frac{n_2 n_1^2}{c^3 m}\right),$$

where

$$\Psi_{0,1}^j(x) = \Psi_0(x) + (-1)^j \frac{\pi^{-3} c^3 m}{n_1^2 n_2 i} \Psi_1(x)$$

and

$$\Psi_k(x) = \int\limits_{\Re s=\sigma} \left(\pi^3 x\right)^{-s} \frac{\Gamma(\frac{1+s+2k+\alpha}{2})\Gamma(\frac{1+s+2k+\beta}{2})\Gamma(\frac{1+s+2k+\gamma}{2})}{\Gamma(\frac{-s-\alpha}{2})\Gamma(\frac{-s-\alpha}{2})\Gamma(\frac{-s-\alpha}{2})} \int\limits_0^\infty \psi(x) x^{-s-k} \frac{\mathrm{d}x}{x}\,\mathrm{d}s. \tag{1}$$

The following, extremely useful lemma is due to Li [5]:

Lemma 2.2. *Suppose $\psi(x)$ is a smooth function compactly supported on $[X, 2X]$ and that $\Psi_0(x)$ is defined as in (1). Then for any fixed integer $K \geqslant 1$ and $xX \gg 1$, we have*

$$\Psi_0(x) = 2\pi^4 xi \int\limits_0^\infty \psi(y) \sum_{j=1}^K \frac{c_j \cos(6\pi x^{1/3} y^{1/3}) + d_j \sin(6\pi x^{1/3} y^{1/3})}{(\pi^3 xy)^{j/3}}\,\mathrm{d}y + O\left((xX)^{\frac{-K+2}{3}}\right),$$

where d_j and c_j are certain constants, specifically, $c_1 = 0$ and $d_1 = -\frac{2}{\sqrt{3\pi}}$.

3. Proof of the main result

Write $\psi(n) = w(n)e(\frac{d^{1/3}n}{M^{2/3}})$. Let us use the Voronoi-type summation formula 2.1:

$$\sum_{M\leqslant n\leqslant M+\Delta} A(1,n)\psi(n) = \frac{\pi^{-5/2}}{4i}\sum_{m>0}\frac{A(m,1)}{m}S(0,m;1)\Psi_{0,1}^0(m) + \frac{\pi^{-5/2}}{4i}\sum_{m>0}\frac{A(m,1)}{m}S(0,-m;1)\Psi_{0,1}^1(m). \tag{2}$$

Now

$$S(0,m;1) = 1 = S(0,-m;1).$$

Therefore, the right side of (2) simplifies to

$$\frac{\pi^{-5/2}}{2i}\sum_{m>0}\frac{A(m,1)}{m}\Psi_0(m).$$

Now we need the asymptotic expansion of $\Psi_0(m)$ (remember that $\psi(x) = w(x)e(\frac{d^{1/3}x}{M^{2/3}})$):

$$\Psi_0(m) = 2\pi^4 mi \int\limits_M^{M+\Delta} w(x) e\left(\frac{d^{1/3}x}{M^{2/3}}\right) \sum_{j=1}^K \frac{c_j \cos(6\pi m^{1/3}x^{1/3}) + d_j \sin(6\pi m^{1/3}x^{1/3})}{(\pi^3 mx)^{j/3}}\,\mathrm{d}x + O\left((mM)^{\frac{-K+2}{3}}\right). \tag{3}$$

Let us first choose K to be the smallest positive integer satisfying the condition

$$\left|A(m,1)\right| \leqslant m^{(K-2)/3-\varepsilon}$$

for some arbitrarily small but fixed $\varepsilon > 0$. Substituting expression (3) into (2) and summing over the error term of (3) yields $\ll M^{(-K+2)/3}$. Let us now write the cosine and sine as sums of exponent functions. Therefore, we may now consider integrals

$$\int\limits_M^{M+\Delta} g_m^{(j)}(x) w(x) e\left(\frac{d^{1/3}x}{M^{2/3}}\right)\mathrm{d}x,$$

where

$$g_m^{(j)}(x) = \sum_{\ell=1}^K a_{\ell,j} e\left((-1)^j 3x^{1/3}m^{1/3}\right)(mx)^{-\ell/3}.$$

Here $a_{\ell,j}$ are some constants; in particular, $a_{1,1} = \frac{-i}{\sqrt{3\pi}}$. Let us only consider the case with $j = 1$ as the other case (for all values of m) can be treated in the same way with the case $j = 1$ and $m \neq d$. Let us first assume $d \neq m$. Using Jutila's and Motohashi's Lemma 6 [4], we obtain

$$\left|\int\limits_M^{M+\Delta} g_m^{(1)}(x) e\left(\frac{d^{1/3}x}{M^{2/3}}\right) w(x)\,\mathrm{d}x\right| \ll \Delta^{-P} m^{-1/3-P/3} M^{-1/3+2P/3}$$

for arbitrarily large but fixed P. Substituting this to (2) yields

$$\sum_{M\leqslant n\leqslant M+\Delta} A(1,n)w(n)e\left(\frac{d^{1/3}n}{M^{2/3}}\right)=\frac{\pi^{-5/2}}{4i}A(d,1)\int_{M}^{M+\Delta} g_d^{(1)}(x)e\left(\frac{d^{1/3}x}{M^{2/3}}\right)w(x)\,\mathrm{d}x+O(1).$$

Let us now consider the case with $d=m$ and $\ell\geqslant 2$. Using integration over absolute values, we obtain

$$\int_{M}^{M+\Delta}\left|a_{\ell,1}w(x)e\left(\frac{d^{1/3}}{M^{2/3}}x-3x^{1/3}d^{1/3}\right)(dx)^{-\ell/3}\right|\mathrm{d}x\ll d^{-\ell/3}\Delta M^{-\ell/3}.$$

Hence,

$$\sum_{M\leqslant n\leqslant M+\Delta} A(1,n)w(n)e\left(\frac{d^{1/3}n}{M^{2/3}}\right)=-\frac{A(d,1)i}{\sqrt{3\pi}d^{-1/3}}\int_{M}^{M+\Delta} w(x)e\left(\frac{d^{1/3}}{M^{2/3}}x-3x^{1/3}d^{1/3}\right)(dx)^{-1/3}\,\mathrm{d}x+O\left(\Delta M^{-2/3}\right). \tag{4}$$

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