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Probability Theory

On some functional of the hybrid process in random scenery

Sur une fonctionnelle du processus hybride en environnement aléatoire

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ABSTRACT

In this work we wish to investigate an example based on the so-called Kesten–Spitzer random walk in random scenery. Namely, replacing the one-dimensional random walk in a general i.i.d. scenery by the hybrids of empirical and partial sums process (see, for instance, [L. Horváth, Approximations for hybrids of empirical and partial sums process, J. Statist. Plann. Inference 88 (2000) 1–18]), we establish an upper bound in the strong approximation for the corresponding functional.

RÉSUMÉ

Dans ce travail nous établissons une borne supérieure dans l'approximation forte d'une fonctionnelle basée sur la marche aléatoire de Kesten–Sptizer en environnement aléatoire, lorsque la marche aléatoire symétrique est remplacée par un processus hybride empirique et des sommes partielles.

1. Introduction

Let $\{\sigma_i\}_{i\in \mathrm{Z}}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables such that

$$E[\sigma_0]=0,\qquad E\big[\sigma_0^2\big]=1\qquad \text{and}\qquad E\big[|\sigma_0|^p\big]<\infty,\quad \text{for all } p>0. \tag{1}$$

Any realisation of the sequence $\{\sigma_i\}_{i\in \mathrm{Z}}$ is called a scenery and let $S=\{S_k\}_{k\in \mathrm{N}}$ be a simple symmetric random walk in Z starting at $S_0=0$. The process $K=\{K_n\}_{n\in \mathrm{N}}$, defined by $K_n=\sum_{0\leqslant k\leqslant n}\sigma(S_k)$, $n\in \mathrm{N}$ is usually referred to as the Kesten–Spitzer random walk in random scenery (RWRS). Some relevant works related to RWRS are for instance (see also references therein), [9] where a continuous analogue for K was introduced and analyzed, [5] where Csáki et al. working on an embedding for the Kesten–Spitzer random walk in random scenery stating also a strong approximation for K_n by a functional corresponding to Brownian motion in Brownian scenery and [3] where Chen and Khoshnevisan prove that model of charged polymers and a model of type K_n are very close to one another.

The aim of this Note is to establish an upper bound in the strong approximation for the functional K_n when S_k is replaced by the process $\{M_n(t),\ 0\leqslant t\leqslant 1\}_{n\geqslant 1}$ defined by

$$M_n(t)=\big[T_n(t)\big]=\Bigg[\sum_{1\leqslant i\leqslant n}\epsilon_i V(Y_i)1_{\{Y_i\leqslant t\}}\Bigg], \tag{2}$$

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$[x]$ denoting the greatest integer satisfying $x-1<[x]\leqslant x$, $\{Y_i,\ 1\leqslant i<\infty\}$ are i.i.d. random variables with common distribution uniform on $[0,1]$, $\{\epsilon_i,\ 1\leqslant i<\infty\}$ are i.i.d. random variables with common distribution uniform given by $P(\epsilon_1=1)=P(\epsilon_1=-1)=1/2$ independent of the Y's, V is a function satisfying $\sup_{t\in[0,1]}|V(t)|\leqslant 1$, and $\{\sigma_i\}_{i\in Z}$ denote a sequence of i.i.d. real-valued random variables satisfying conditions given in (1). In this case we will be concerned with the approximation of the functional given by

$$K_n(t)=\sum_{l=0}^{n}\sigma\big(M_l(t)\big)=\sum_{x\in Z}\sigma(x)L_t^x(M_n), \tag{3}$$

by a functional corresponding to Brownian motion in Brownian scenery (BMRS) (see $g_n(t)$, Section 2). The second equality in (3) gives us that K_n can be represented compactly (see for instance [3,5]) where $L_t^x(M_n)$ is the local time of the process M_n which we will define in (5).

Our main result and the proof will be given in Section 2. Without loss of generality, we will assume that all the random variables and the stochastic processes introduced throughout are defined on the same probability space.

Let us now recall some facts related to the processes that are in studying. The process T_n given in (2) can be obtained from the process $\tilde{T}_n$ defined by $\tilde{T}_n(t)=\sum_{1\leqslant i\leqslant n}\epsilon_i H(X_i)1_{\{X_i\leqslant t\}}$, $-\infty<t<\infty$ where the sequence $\{X_i, 1\leqslant i<\infty\}$ are i.i.d. random variables with common distribution function F, independent of the sequence $\{\epsilon_i, 1\leqslant i<\infty\}$ and the function H is positive and has bounded variation on the real line. By [8, p. 5], we have without loss of generality, there are i.i.d. random variables $\{Y_i,\ 1\leqslant i<\infty\}$ uniform on $[0,1]$ such that $X_i=Q(Y_i)$, with $Q(y)=\inf\{x:\ F(x)\geqslant y\}$, i.e., the quantile function of F, then we can consider $T_n(t)$ in the place of $\tilde{T}_n(t)$.

Let B be a standard Wiener process. The jointly continuous version of the local time of B, is usually defined by the occupation time formula: for all Borel non-negative function f,

$$\int_0^t f\big(B(s)\big)\,ds=\int_R f(x)L_t^x(B)\,dx, \quad t>0. \tag{4}$$

The local time of the process $M_n=\{[\sum_{i=1}^n \epsilon_i V(Y_i)1_{\{Y_i\leqslant t\}}],\ t\in[0,1]\}$, is given by

$$L_t^x(M_n)=\frac{1}{\sqrt{n}}\sum_{s\leqslant t}1_{\{M_n(s)=x\}}, \quad t\in[0,1],\ x\in Z. \tag{5}$$

Remark that the local time of M_n denoted by $L_t^x(M_n)$ corresponds to $\int_x^{x+1}L_t^y(T_n)\,dy$ (i.e. the occupation time of $[x,x+1)$ for the process T_n). Now, in the same spirit of [1], denoting $U_n(t)=\sum_{i=1}^{[nt]}\epsilon_i$ for $0\leqslant t\leqslant 1$ and $N_n(t)=\sum_{1\leqslant i\leqslant n}V(Y_i)1_{\{Y_i\leqslant t\}}/n$ for $0\leqslant t\leqslant 1$, we define the local time of the process T_n by

$$L_t^x(T_n)=\int_0^1 L_t^s(N_n)\,d_s L_s^x(U_n) \quad (x\in Z).$$

2. Results and proofs

Here and subsequently, we consider a sequence of independent Brownian motions W_{1n} and an independent Brownian motion W_2, where $W_{1n}(t)=W_1(nG_n(t))$ with $G_n(t)=\int_0^t V^2(s)\,dE_n(s)$ and $E_n(t)$ is the empirical (uniform) distribution function based on the first n terms of the sequence of random variables Y_i's. We consider also the associated functional called the Brownian motion in Brownian scenery defined by $g_n(t)=\int_R L_t^x(W_{1n})\,dW_2(x)$ where $L_t^x(W_{1n})$ is the local time of the process W_{1n}. Let X be a process we define $L_1^x(X)=\sup_{0\leqslant t\leqslant 1}L_t^x(X)$.

In our first result we establish an upper bound for the approximation of the local time of the process M_n by the local time of W_{1n} given by $L_1^x(W_{1n})=\int_0^1 L_1^s(G_n)\,ds\,L_s^x(W_n)$, this local was defined on [1] in the light of the strong approximation given by Diebolt in [7].

Theorem 2.1. *Under conditions related to the definition of the process M_n. Then we can define a sequence of Wiener processes $\{W_{1n}(t),\ 0\leqslant t\leqslant 1\}$ such that*

$$\sup_{x\in Z}\big|L_1^x(M_n)-L_1^x(W_{1n})\big|=o\big(n^{1/4+\eta}\big), \quad a.s.$$

Our next theorem is the main result:

Theorem 2.2. *Under conditions given in* (1) *and under conditions of Theorem* 2.1, *we get*

$$\left|\sum_{x\in Z} L_1^x(M_n)\sigma_x - \int_R L_1^x(W_{1n})\,dW_2(x)\right| = o\big(n^{5/8+\epsilon}\big), \quad a.s.$$

where W_2 is a Wiener processes independent of W_{1n} and defined in the same probability space.

Proof of Theorem 2.1. The first part of our proof starts with an overview of the arguments given in [1] useful for to state some properties of the local time of the process T_n. Let $\{H_n(t)\}_{t\geqslant 0}$ be a compensated compound Poisson process. We have that $\{\alpha_n(t);\ t\geqslant 0\} \equiv \{H_n(t);\ t\geqslant 0 \mid H_n(1)=0\}$, where $\alpha_n(t)$ is the uniform empirical process and $H_n(t) = H(nt)/\sqrt{n}$ with $\{H(nt)\}_{n\geqslant 1}$ is a sequence of compensated Poisson process with expected arrival rate of $1/n$. By using crossing comparison (see p. 339 of [10]) we obtain that $\sqrt{n}H_n(s) = H(ns)$, then we can consider that $L_t^x(N_n) = L_t^{x'}(H_n)$ for $x\in R$ and $x' = x - n\int_0^t V^2(s)\,ds$. In the same way as in the proof of Proposition 1 of [1], we get

$$\sup_{x,y\in R:\ |x-y|\leqslant b_n} \big|L_1^x(T_n) - L_1^x(W_{1n})\big| = O\big(n^{1/4+\delta'}\big), \quad \text{a.s.} \tag{6}$$

for some $\delta' > 0$ and $b_n = O(n^{-1/2})$ (see Lemma 3.1 in [2] and Fact 7.2 in [6, p. 1052]). In the second part of the proof, we state the upper bound in the approximation of the local time of M_n denoted by $L_t^x(M_n)$. Remember that $L_t^x(M_n)$ corresponds to $\int_x^{x+1} L_t^y(T_n)\,dy$. Then we have

$$\sup_{x\in Z}\big|L_1^x(M_n) - L_1^x\big([W_{1n}]\big)\big| \leqslant I_1 + I_2 + I_3,$$

where

$$I_1 = \sup_{x\in Z}\left|\int_x^{x+1}\big(L_1^y(T_n) - L_1^x(T_n)\big)\,dy\right|, \qquad I_2 = \sup_{x\in Z}\big|L_1^x(T_n) - L_1^x(W_{1n})\big| \qquad \text{and}$$

$$I_3 = \sup_{x\in Z}\left|\int_x^{x+1}\big(L_1^y(W_{1n}) - L_1^x\big([W_{1n}]\big)\big)\,dy\right|.$$

By using (6) we have that $I_2 = O(n^{1/4+\delta'})$, a.s. By using (2.19) of [4], we have $I_3 = O(n^{1/4+\eta_1})$, a.s.

Finally, replacing T_n by W_{1n} in $|L_1^y(T_n) - L_1^x(T_n)|$ and by making use of (2.19) of [4] and (6), we get that $I_1 = O(n^{1/4+\eta})$. It is easily seen that the announced rate is obtained from those of I_1, I_2 and I_3. The proof of Theorem 2.1 is complete. □

Remark 1. By using the law of the iterated logarithm (LIL) for W_{1n}, we have that $L_t^x(W_{1n}) = 0$, a.s. for $x > \sqrt{2n\log_2 n}$ and in the same way, we have that $L_t^x(T_n) = 0$, a.s. for $x > \sqrt{2n\log_2 n}$.

Proof of Theorem 2.2. The proof will be based on two propositions giving some useful results.

Proposition 2.3. *Under the same conditions of Theorem 2.2, we get*

$$I_n = \left|\sum_{x\in Z}\sigma_x\big(L_1^x(M_n) - L_1^x(W_{1n})\big)\right| = O\big(n^{1/2+\eta}\big), \quad a.s.$$

Proof. By using (6) we get $I_n = |\sum_{x\in Z}\sigma_x| O(n^{1/4+\delta'})$, a.s. By using the LIL for $|\sum_{x=-N}^N \sigma_x| = O(\sqrt{N\log_2 N})$. Now, by using Remark 1 we have that $N = [\sqrt{n\log_2 n}]$ then $|\sum_{x=-N}^N \sigma_x| = O(n^{1/4+\eta_2})$. It is now a direct consequence of the above results the announced upper bound.

The following proposition is given without proof, because is in the same vein of Proposition 2.2 of [5]:

Proposition 2.4. *Under the same conditions as in Theorem 2.2, we have*

$$\left|\sum_{x\in Z}\sigma_x L_1^x(W_{1n}) - \int_R L_1^x(W_{1n})\,dW_2(x)\right| = o\big(n^{5/8+\epsilon}\big), \quad a.s.$$

The proof of Theorem 2.2 is a direct consequence of Propositions 2.1 and 2.2.

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