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Optimal double stopping time problem

Arrêt optimal avec deux temps d'arrêt

Magdalena Kobylanski [a], Marie-Claire Quenez [b], Elisabeth Rouy-Mironescu [c]

[a] *LAMA, CNRS – UMR 8050, université Paris Est, cité Descartes, 5, boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France*
[b] *LPMA, CNRS – UMR 7599, université Paris-Diderot, 175 rue du Chevaleret, 75013 Paris, France*
[c] *ICJ, École centrale de Lyon, CNRS – UMR 5208, 36, avenue Guy-de-Collongue, 69134 Ecully cedex, France*

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ABSTRACT

We consider the optimal double stopping time problem $v(S) = \operatorname{ess\,sup}\{E[\psi(\tau_1, \tau_2) \mid \mathcal{F}_S],\ \tau_1, \tau_2 \geqslant S\}$ for each stopping time S. Following the optimal one stopping time problem, we study the existence of optimal stopping times and give a method to compute them. The key point is the construction of a new reward ϕ such that $v(S) = \operatorname{ess\,sup}\{E[\phi(\tau) \mid \mathcal{F}_S],\ \tau \geqslant S\}$.

RÉSUMÉ

Nous étudions le problème d'arrêt optimal avec deux temps d'arrêt où la fonction valeur est définie par $v(S) = \operatorname{ess\,sup}\{E[\psi(\tau_1, \tau_2) \mid \mathcal{F}_S],\ \tau_1, \tau_2 \geqslant S\}$ pour chaque temps d'arrêt S. Nous montrons que ce problème se réduit à un problème d'arrêt optimal avec un seul temps d'arrêt pour un nouveau rendement ϕ, *i.e.* $v(S) = \operatorname{ess\,sup}\{E[\phi(\tau) \mid \mathcal{F}_S],\ \tau \geqslant S\}$. Nous montrons l'existence de temps d'arrêt optimaux sous des hypothèses de régularité sur ψ.

Version française abrégée

Soit $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, P)$ un espace probabilisé où $T > 0$ est le temps terminal et où la filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ satisfait les hypothèses habituelles. On suppose que $\mathcal{F}_0$ contient tous les mesurables de probabilité 0 ou 1. On note T_0 l'ensemble des temps d'arrêt à valeurs dans $[0, T]$. Plus généralement, pour tout temps d'arrêt S, on note T_S l'ensemble des temps d'arrêt $\theta \in T_0$ tels que $S \leqslant \theta$ p.s. De plus, on note $t_n \uparrow t$ si (t_n) est une suite croissante telle que $\lim_{n \to \infty} t_n = t$.

Définition 0.1. On dit que la famille $\{\psi(\theta, S),\ \theta, S \in T_0\}$ est *biadmissible* si pour tout $\theta, S \in T_0$, $\psi(\theta, S)$ est une variable aléatoire (v.a.) positive $\mathcal{F}_{\theta \vee S}$-mesurable et si pour tout $\theta, \theta', S, S' \in T_0$, $\psi(\theta, S) = \psi(\theta', S')$ p.s. sur $\{\theta = \theta'\} \cap \{S = S'\}$.

Soit $\{\psi(\theta, S),\ \theta, S \in T_0\}$ une famille biadmissible appelée rendement. La fonction valeur associée est définie pour chaque temps d'arrêt S par :

$$v(S) = \operatorname*{ess\,sup}_{\tau_1, \tau_2 \in T_S} E\big[\psi(\tau_1, \tau_2) \mid \mathcal{F}_S\big]. \tag{1}$$

E-mail addresses: magdalena.kobylanski@univ-mlv.fr (M. Kobylanski), quenez@math.jussieu.fr (M.-C. Quenez), Elisabeth.Mironescu@ec-lyon.fr (E. Rouy-Mironescu).

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Pour chaque temps d'arrêt θ, on définit les v.a. $\mathcal{F}_\theta$-mesurable :

$$u_1(\theta) = \operatorname*{ess\,sup}_{\tau_1 \in T_\theta} E\big[\psi(\tau_1, \theta) \mid \mathcal{F}_\theta\big], \qquad u_2(\theta) = \operatorname*{ess\,sup}_{\tau_2 \in T_\theta} E\big[\psi(\theta, \tau_2) \mid \mathcal{F}_\theta\big]. \tag{2}$$

On définit maintenant le nouveau rendement :

$$\phi(\theta) = \max\big[u_1(\theta), u_2(\theta)\big], \tag{3}$$

et pour $S \in T_0$ on note $u(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\phi(\theta) \mid \mathcal{F}_S]$ la fonction valeur associée.

Théorème 0.2 *(Reduction). Pour tout temps d'arrêt S, $v(S) = u(S)$ p.s.*

Afin d'obtenir un résultat d'existence de temps d'arrêt optimaux, on introduit la définition suivante :

Définition 0.3. On dit que $\{\psi(\theta, S),\ \theta, S \in T_0\}$ est *uniformément continue en espérance le long des temps d'arrêt* si pour tout $\theta, S \in T_0$ et pour toute suite monotone (S_n) de temps d'arrêt tendant vers S p.s. on a $\lim_{n\to\infty} E(\operatorname{ess\,sup}_{\theta \in T_0} |\psi(\theta, S) - \psi(\theta, S_n)|) = 0$ et $\lim_{n\to\infty} E(\operatorname{ess\,sup}_{\theta \in T_0} |\psi(S, \theta) - \psi(S_n, \theta)|) = 0$.

Théorème 0.4. *Soit $\{\psi(\theta, S),\ \theta, S \in T_0\}$ une famille biamissible uniformément continue en espérance le long des temps d'arrêt. Supposons qu'il existe $p > 1$ tel que $E[\operatorname{ess\,sup}_{\theta, S \in T_0} \psi(\theta, S)^p] < \infty$.*

Alors, il existe une paire (τ_1^*, τ_2^*) de temps d'arrêt optimaux pour $v(S)$.

1. The optimal one stopping time problem revisited

Recall the notion of admissible family of random variables introduced by N. El Karoui (see [1]).

Definition 1.1. We say that a family $\{\phi(\theta),\ \theta \in T_0\}$ is *admissible* if for all $\theta \in T_0$ $\phi(\theta)$ is a $\mathcal{F}_\theta$-measurable positive random variable (r.v.), and for all $\theta, \theta' \in T_0$, $\phi(\theta) = \phi(\theta')$ on $\{\theta = \theta'\}$.

Classically, in optimal stopping time problems (see for example [2]), the reward is given by a RCLL adapted process (ϕ_t). Note that in this case, the family of r.v. defined by $\{\phi(\theta) = \phi_\theta,\ \theta \in T_0\}$ is admissible.

We will show in the present work that computing the value function for the optimal double stopping time problem reduces to computing the value function for an optimal one stopping time problem where the *new reward* ϕ is no longer a RCLL process but an admissible family $\{\phi(\theta), \theta \in T_0\}$. Actually, this notion of admissible family of r.v. is essential to study the optimal double stopping time problem.

Let $\{\phi(\theta),\ \theta \in T_0\}$ be a given admissible family (called *reward*). The *value function at time S* associated with reward $\{\phi(\theta),\ \theta \in T_0\}$ is given at each time $S \in T_0$ by $v(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\phi(\theta) \mid \mathcal{F}_S]$.

Recall that the family of r.v. $\{v(S),\ S \in T_0\}$ is admissible and is a *supermartingale* system, that is for any stopping times $\theta, \theta' \in T_S$ such that $\theta \geqslant \theta'$, $E[v(\theta) \mid \mathcal{F}_{\theta'}] \leqslant v(\theta')$ a.s.

Definition 1.2. An admissible family $\{\phi(\theta),\ \theta \in T_0\}$ is said to be *right* (*resp. left*) *continuous along stopping times in expectation* (RCE (resp. LCE)) if for any $\theta \in T_0$ and for any $\theta_n \downarrow \theta$ a.s. (resp. $\theta_n \uparrow \theta$ a.s.) one has $E[\phi(\theta)] = \lim_{n\to\infty} E[\phi(\theta_n)]$.

Recall the following classical lemma (see El Karoui [1]):

Lemma 1.3. *Let $\{\phi(\theta),\ \theta \in T_0\}$ be an admissible family RCE such that $E[\operatorname{ess\,sup}_{\theta \in T_0} \phi(\theta)] < \infty$. Then, the family $\{v(S),\ S \in T_0\}$ is RCE.*

We will now state a theorem which generalizes the classical existence result of optimal stopping times to the case of a reward given by an admissible family of r.v. (instead of a RCLL adapted process). For each $S \in T_0$, let us introduce the following $\mathcal{F}_S$-measurable random variable $\theta^*(S)$ defined by

$$\theta^*(S) := \operatorname{ess\,inf}\big\{\theta \in T_S,\ v(\theta) = \phi(\theta) \text{ a.s.}\big\}. \tag{4}$$

Note that $\theta^*(S)$ is a stopping time. Indeed, for each $S \in T_0$, one can easily show that the set $\mathcal{T}_S = \{\theta \in T_S,\ v(\theta) = \phi(\theta)$ a.s.$\}$ is closed under pairwise maximization. By a classical result (see Neveu [5]), there exists a sequence $(\theta^n)_{n\in\mathbb{N}}$ of stopping times in $\mathcal{T}_S$ such that $\theta_n \downarrow \theta^*(S)$ a.s. Furthermore, we have

Theorem 1.4. *Let $\{\phi(\theta),\ \theta \in T_0\}$ be an admissible family RCE and LCE such that $E[\operatorname{ess\,sup}_{\theta \in T_0} \phi(\theta)] < \infty$. Let $\{v(S),\ S \in T_0\}$ be the associated value function family. Then, for each $S \in T_0$, the stopping time $\theta^*(S)$ defined by* (4) *is optimal for $v(S)$ (i.e. $v(S) = E[\phi(\theta^*(S)) \mid \mathcal{F}_S]$).*

As in the case of a reward process, the proof is based on a penalization method.

2. The optimal double stopping time problem

We consider now the optimal double stopping time problem. Let us introduce the following definitions:

Definition 2.1. The family $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is a *biadmissible family* if for all $\theta, S \in T_0$, $\psi(\theta, S)$ is a $\mathcal{F}_{\theta\vee S}$-measurable positive random variable and if for all $\theta, \theta', S, S' \in T_0$, $\psi(\theta, S) = \psi(\theta', S')$ a.s. on $\{\theta = \theta'\} \cap \{S = S'\}$.

Let $\{\psi(\theta, S),\ \theta, S \in T_0\}$ be a *biadmissible family* called reward. The value function associated with this reward is defined for each stopping time S by

$$v(S) = \operatorname*{ess\,sup}_{\tau_1,\tau_2 \in T_S} E\big[\psi(\tau_1, \tau_2) \mid \mathcal{F}_S\big]. \tag{5}$$

As in the case of one stopping time problem, the family $\{v(S),\ S \in T_0\}$ is an admissible family of positive r.v.

2.1. Reduction to an optimal one stopping time problem

Recall that this optimal double stopping time problem can be expressed in terms of one stopping time problems as follows. Note that since $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is admissible, the families $\{u_1(\theta),\ \theta \in T_S\}$ and $\{u_2(\theta),\ \theta \in T_S\}$ defined by (2) are admissible. Moreover, the new reward $\{\phi(\theta),\ \theta \in T_S\}$ defined by (3) is also clearly admissible. Note that the value function associated with the new reward is given by $u(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\phi(\theta) \mid \mathcal{F}_S]$ a.s.

Theorem 2.2 *(Reduction). Suppose that $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is a biadmissible family. For each stopping time S, consider the associated value function $v(S)$ defined by* (5) *and $u(S)$ the value function associated with the new reward ϕ. Then, $v(S) = u(S)$ a.s.*

Proof. Let S be a stopping time.

Step 1. Let us show first that $v(S) \leqslant u(S)$ a.s. Let $\tau_1, \tau_2 \in T_S$. Put $A = \{\tau_1 \leqslant \tau_2\}$. As A is in $\mathcal{F}_{\tau_1\wedge\tau_2}$,

$$E\big[\psi(\tau_1, \tau_2) \mid \mathcal{F}_S\big] = E\big[\mathbf{1}_A E\big[\psi(\tau_1, \tau_2) \mid \mathcal{F}_{\tau_1\wedge\tau_2}\big] \mid \mathcal{F}_S\big] + E\big[\mathbf{1}_{A^c} E\big[\psi(\tau_1, \tau_2) \mid \mathcal{F}_{\tau_1\wedge\tau_2}\big] \mid \mathcal{F}_S\big].$$

By noticing that on A one has $E[\psi(\tau_1, \tau_2) \mid \mathcal{F}_{\tau_1\wedge\tau_2}] = E[\psi(\tau_1, \tau_2) \mid \mathcal{F}_{\tau_1}] \leqslant u_2(\tau_1) \leqslant \phi(\tau_1 \wedge \tau_2)$ a.s., and similarly on A^c one has $E[\psi(\tau_1, \tau_2) \mid \mathcal{F}_{\tau_1\wedge\tau_2}] = E[\psi(\tau_1, \tau_2) \mid \mathcal{F}_{\tau_2}] \leqslant u_1(\tau_2) \leqslant \phi(\tau_1 \wedge \tau_2)$ a.s., we get $E[\psi(\tau_1, \tau_2) \mid \mathcal{F}_S] \leqslant E[\phi(\tau_1 \wedge \tau_2) \mid \mathcal{F}_S] \leqslant u(S)$ a.s.

By taking the supremum over τ_1 and τ_2 in T_S we obtain step 1.

Step 2. We will now show that $v(S) \geqslant u(S)$ a.s. To simplify, suppose the existence of optimal stopping times for the three one optimal stopping time problems. Let θ^* be an optimal stopping time for $u(S)$, *i.e.* $u(S) = E[\phi(\theta^*) \mid \mathcal{F}_S]$ a.s. and let θ_1^* and θ_2^* be optimal stopping times for $u_1(\theta^*)$ and $u_2(\theta^*)$, *i.e.* $u_1(\theta^*) = E[\psi(\theta_1^*, \theta^*) \mid \mathcal{F}_{\theta^*}]$ a.s. and $u_2(\theta^*) = E[\psi(\theta^*, \theta_2^*) \mid \mathcal{F}_{\theta^*}]$ a.s.

Put $B = \{u_1(\theta^*) \leqslant u_2(\theta^*)\}$. Let (τ_1^*, τ_2^*) be the stopping times defined by

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}; \qquad \tau_2^* = \theta_2^* \mathbf{1}_B + \theta^* \mathbf{1}_{B^c}, \tag{6}$$

as B is $\mathcal{F}_{\theta^*}$ measurable, and since $u(S) = E[\phi(\theta^*) \mid \mathcal{F}_S]$, we have $u(S) = E[\mathbf{1}_B u_2(\theta^*) + \mathbf{1}_{B^c} u_1(\theta^*) \mid \mathcal{F}_S]$.

Hence, we have $u(S) = E[\mathbf{1}_B \psi(\theta^*, \theta_2^*) + \mathbf{1}_{B^c} \psi(\theta_1^*, \theta^*) \mid \mathcal{F}_S] = E[\psi(\tau_1^*, \tau_2^*) \mid \mathcal{F}_S]$.

Hence, $u(S) \leqslant v(S)$ a.s. Note that in the general case, when we do not suppose the existence of optimal stopping times, this inequality can be obtained by using optimizing sequences of stopping times. □

Remark 1. Note that step 2 gives a method to construct a pair of optimal stopping times (τ_1^*, τ_2^*) for $v(S)$.

2.2. Existence of optimal stopping times

Before studying the problem of existence of optimal stopping times, we have to state some regularity properties of the new reward family $\{\phi(\theta),\ \theta \in T_0\}$. Let $\{\psi(\theta, S),\ \theta, S \in T_0\}$ be a biadmissible family such that $E[\operatorname{ess\,sup}_{\theta,S\in T_0} \psi(\theta, S)] < \infty$. Let us introduce the following definition:

Definition 2.3. The family $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is said to be *uniformly right* (*resp. left*) *continuous in expectation along stopping times* (URCE (resp. ULCE)) if for each θ, $S \in T_0$ and each sequence (S_n) of T_0 such that $S_n \downarrow S$ a.s. (resp. $S_n \uparrow S$ a.s.) we have

$$\lim_{n\to\infty} E\Big(\operatorname*{ess\,sup}_{\theta\in T_0} \big|\psi(\theta, S) - \psi(\theta, S_n)\big|\Big) = 0 \quad \text{and} \quad \lim_{n\to\infty} E\Big(\operatorname*{ess\,sup}_{\theta\in T_0} \big|\psi(S, \theta) - \psi(S_n, \theta)\big|\Big) = 0.$$

The following continuity property holds true for the new reward family:

Theorem 2.4. *Suppose that the biadmissible family $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is URCE and ULCE. Then, the family $\{\phi(S),\ S \in T_0\}$ defined by* (3) *is RCE and LCE.*

Proof. Let us show the RCE property. As $\phi(\theta) = \max[u_1(\theta), u_2(\theta)]$, it is sufficient to show the RCE property for the family $\{u_1(\theta),\ \theta \in T_0\}$. Let us introduce the following value function for each $S, \theta \in T_0$, $U_1(\theta, S) = \operatorname{ess\,sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, S) \mid \mathcal{F}_\theta]$ a.s.

As for all $\theta \in T_0$, $u_1(\theta) = U_1(\theta, \theta)$ a.s. it is sufficient to prove that $\{U_1(\theta, \theta),\ \theta \in T_0\}$ is RCE.

Let $\theta \in T_0$ and $(\theta_n)_n$ be a sequence of stopping times such that $\theta_n \downarrow \theta$ a.s. We have $|E[U_1(\theta, \theta)] - E[U_1(\theta_n, \theta_n)]| \leqslant |E[U_1(\theta, \theta)] - E[U_1(\theta_n, \theta)]| + |E[U_1(\theta_n, \theta)] - E[U_1(\theta_n, \theta_n)]|$.

The first term of the right hand side converges a.s. to 0 as n tends to ∞ by the RCE property of the value function $\{U_1(\tau, \theta),\ \tau \in T_0\}$ by Lemma 1.3. The second term converges to 0 since the family ψ is URCE. The LCE property of the new reward follows by similar arguments. □

We now state the following existence result of optimal stopping times:

Theorem 2.5. *Let $\{\psi(\theta, S),\ \theta, S \in T_0\}$ be a biadmissible family which is URCE and ULCE. Suppose that there exists $p > 1$ such that $E[\operatorname{ess\,sup}_{\theta, S \in T_0} \psi(\theta, S)^p] < \infty$. Then, there exists a pair (τ_1^*, τ_2^*) of optimal stopping times for $v(S)$ defined by* (5)*, that is $v(S) = E[\psi(\tau_1^*, \tau_2^*) \mid \mathcal{F}_S]$.*

Proof. By Theorem 2.4 the admissible family of positive r.v. $\{\phi(\theta),\ \theta \in T_0\}$ defined by (3) is RCE and LCE. By Theorem 1.4 the stopping time $\theta^*(S) = \operatorname{ess\,inf}\{\theta \in T_S \mid v(\theta) = \phi(\theta) \text{ a.s.}\}$ is optimal for $u(S) = v(S)$, that is $u(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\phi(\theta) \mid \mathcal{F}_S] = E[\phi(\theta^*) \mid \mathcal{F}_S]$ a.s. Moreover, the families $\{\psi(\theta, \theta^*),\ \theta \in T_{\theta^*}\}$ and $\{\psi(\theta^*, \theta),\ \theta \in T_{\theta^*}\}$ are clearly admissible and are RCE and LCE. Consider the following optimal stopping time problem:

$$v_1(S) = \operatorname*{ess\,sup}_{\theta \in T_S} E\big[\psi\big(\theta, \theta^*\big) \mid \mathcal{F}_S\big] \quad \text{and} \quad v_2(S) = \operatorname*{ess\,sup}_{\theta \in T_S} E\big[\psi\big(\theta^*, \theta\big) \mid \mathcal{F}_S\big].$$

Let θ_1^* and θ_2^* are optimal stopping times given by Theorem 1.4 for $v_1(\theta^*)$ and $v_2(\theta^*)$.

Note that $v_1(\theta^*) = u_1(\theta^*)$ and $v_2(\theta^*) = u_2(\theta^*)$.

The existence of optimal stopping times (τ_1^*, τ_2^*) for problem (5) follows by Remark 1. □

Remark 2. Theorems 2.2 and 2.5 can be generalized by induction to the case of d stopping times (see Kobylanski et al. [4]).

2.3. Example: an exchange option

Suppose that the market contains two risky assets with price processes $X_t = (X_t^1, X_t^2)$ which satisfy $dX_t^1 = rX_t^1\,dt + \sigma_1 X_t^1\,dW_t^1$ and $dX_t^2 = rX_t^2\,dt + \sigma_2 X_t^2\,dW_t^2$, where $W = (W^1, W^2)$ is a $\mathbb{R}^2$-valued Brownian motion and $(\mathcal{F}_t)_{t \geqslant 0}$ is the associated standard filtration. Without loss of generality, we first suppose that the interest rate r is equal to 0 (the general case can be derived by considering discounted prices). The buyer of the option chooses two stopping times τ_1 (for the first asset) and τ_2 (for the second one) smaller than T. At time $\tau_1 \vee \tau_2$, he is allowed to exchange the first asset against the second one; in other terms, he receives the amount: $(X_{\tau_1}^1 - X_{\tau_2}^2)^+$.

Given the initial data $(s, x) = (s, x_1, x_2) \in [0, T] \times \mathbb{R}^2$, the price at time s of this option is given by

$$v(s, x) = \operatorname*{ess\,sup}_{\tau_1, \tau_2 \in T_s} E\big[\big(X_{\tau_1}^{1,s,x_1} - X_{\tau_2}^{2,s,x_2}\big)^+ \mid \mathcal{F}_s\big]. \tag{7}$$

Given initial data $(s, x_1) \in [0, T] \times \mathbb{R}$ consider also $v^1(s, x_1, y_2) = \operatorname{ess\,sup}_{\tau_1 \in T_s} E[(X_{\tau_1}^{1,s,x_1} - y_2)^+ \mid \mathcal{F}_s]$, for each fixed parameter $y_2 \in \mathbb{R}$. We have clearly that $v^1(s, x_1, y_2) = y_2 C_1(s, \frac{x_1}{y_2})$, where for initial data $(s, z) \in [0, T] \times \mathbb{R}$, $C_1(s, z) = E[(X_T^{1,s,z} - 1)^+ \mid \mathcal{F}_s]$, which is explicitly given by the Black–Scholes formula. Similarly, given initial data $(t, y_2) \in [0, T] \times \mathbb{R}$, consider $v^2(x_1, t, y_2) = \operatorname{ess\,sup}_{\tau_2 \in T_t} E[(x_1 - X_{\tau_2}^{2,t,y_2})^+ \mid \mathcal{F}_t]$ for each fixed parameter $x_1 \in \mathbb{R}$. We have clearly that $v^2(x_1, t, y_2) = x_1 P_2(t, \frac{y_2}{x_1})$, where for initial data $(s, z) \in [0, T] \times \mathbb{R}$, $P_2(t, z) = E[(1 - X_T^{2,t,z})^+ \mid \mathcal{F}_t]$. Note that the function $P_2(s, z)$ corresponds to the price of an European put option. We set

$$\phi(s, x_1, x_2) = \max\left(x_2 C_1\left(s, \frac{x_1}{x_2}\right), x_1 P_2\left(s, \frac{x_2}{x_1}\right)\right), \quad \text{for } s, x_1, x_2 \in [0, T] \times \mathbb{R}^2.$$

By the previous results, the price function v is the value function of the following optimal stopping time problem: $v(s, x_1, x_2) = \operatorname{ess\,sup}_{\theta \in T_s} E[\phi(\theta, X_\theta^{1,s,x_1}, X_\theta^{2,s,x_2}) \mid \mathcal{F}_s]$. It corresponds to the price of an American option with maturity T and payoff $(\phi(t, X_t^1, X_t^2))_{0 \leqslant t \leqslant T}$. By classical results, it is a viscosity solution of an obstacle problem. By Remark 1, we can easily construct a pair (τ_1^*, τ_2^*) of optimal stopping times for the exchange option (7). Using some results of Oksendal and Sulem [6], some examples in the case of jump diffusions can also be studied (see Kobylanski and Quenez [3]).

References

[1] N. El Karoui, Les Aspects Probabilistes du Contrôle Stochastique, in : Ecole d'été de Probabilités de Saint-Flour IX, Springer-Verlag, 1981, p. 876.
[2] I. Karatzas, S. Shreve, Methods of Mathematical Finance, Springer-Verlag, 1999.
[3] M. Kobylanski, M.-C. Quenez, Optimal multiple stopping: the Markovian case and applications to finance, working paper, 2009.
[4] M. Kobylanski, M.-C. Quenez, E. Rouy-Mironescu, Optimal multiple stopping time problem, arXiv:0910.2788v1.
[5] J. Neveu, Discrete-Parameter Martingales, English translation, North-Holland/American Elsevier, Amsterdam/New York, 1975.
[6] B. Oksendal, A. Sulem, Applied Stochastic Control of Jump Diffusions, Springer, 2007.