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Theoretical analysis for the deflection of granular jets

Analyse théorique de la déflection de jets granulaires

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ABSTRACT

We consider a dilute granular jet colliding with an oblique planar wall. Collisions between particles and collisions between particles and the wall are inelastic. We derive an exact solution for the mean force experienced by the wall for dilute jets. We show that the mean force on the wall can be a non-monotonic function of the angle between the wall and the jet. This occurs because particles that rebound from the wall can collide with incoming particles and be scattered.

RÉSUMÉ

On considere un jet granulaire dilué qui entre en collision avec un mur plan oblique. Les collisions entre particules ainsi que les collisions mur-particules sont inélastiques. Nous obtenons une solution exacte pour la force moyenne supportée par le mur pour les jets dilués. Nous montrons que la force moyenne peut être une fonction non monotone de l'angle entre la paroi et le jet du fait de la diffusion des particules qui, après avoir rebondi sur le mur, s'entrechoquent avec les particules entrantes.

1. Introduction

In this Note we consider the mechanisms that underlie the process of deflection when a granular jet collides with a rigid wall. By a granular jet, we mean a localized stream of discrete particles moving with the same velocity. The collisions between particles and the collisions between particles and the wall are inelastic. We will show that this seemingly simple system can give rise to an array of surprising dynamics.

Hákonardóttir and Hogg [2] considered the interaction of granular flows with deflecting dams. They performed experimental studies and developed a theoretical framework to describe free-surface flows. In bidisperse particle systems, the case of shocks induced by a moving boundary was considered by Wylie et al. [4]. Wylie and Zhang [3] showed that phase-locking and complicated orbits collapse occur for dissipative particle systems that are driven by forcing from a boundary. Wylie et al. [6] and Wylie et al. [5] studied the motion of a large number of particles in a closed box that are excited by a vibrating boundary and experience a linear drag force from the interstitial fluid. In this Note, we will focus on the effective force experienced by the rigid boundary and show that a number of surprising phenomena can occur.

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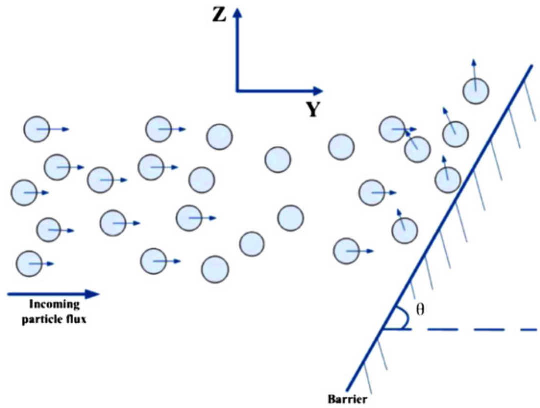

Fig. 1. Schematic (Schéma).

2. Formulation

We consider a system in which a dilute granular jet originating from infinity collides with an oblique planar wall of infinite length (see Fig. 1). We assume that the jet consists of identical smooth spheres of radius a and mass m. We further assume that all incoming particles have the same velocity v_0. The angle between the wall and the direction of the jet is θ. We will consider a 2-dimensional system, but the methodology explained in this Note can be extended to 3-dimensions in a straightforward way. We define axes with the Y-axis parallel to the jet, and the Z-axis perpendicular to the jet.

Collisions between particles are inelastic with a constant restitution coefficient $0 \leqslant e \leqslant 1$, which is defined as the ratio of the relative velocities of two particles in the direction along their line of centers immediately after and immediately before the particle–particle collision. The coefficient of restitution is assumed to be independent of velocity and denotes the degree of dissipation in particle collisions. Similarly, the collisions between particles and the wall are characterized by the restitution coefficient $0 \leqslant e_W \leqslant 1$.

The particles in the jet are randomly located, and in general, the spatial distributions of particles in the directions parallel and perpendicular to the jet will be different. Without loss of generality, we choose the coordinate system such that the mean Z-location of particles is zero. We denote the marginal density function of the Z-location of particles as ρ_Z and the standard deviation of ρ_Z as σ_Z. We denote the marginal density function of the distance in the Y-direction between adjacent particles as ρ_Y and the mean of ρ_Y as μ_Y. We note that for dense jets in which $\mu_Y \ll a$ there will be strong dependence between the locations of particles since that two particles cannot occupy the same location. However, for the relatively dilute jets which we will consider in this Note, the dependence will be weak. Here we choose ρ_Z to be a Gaussian distribution with standard deviation σ_Z. For simplicity, we choose ρ_Y to be non-random, that is all particles are spaced μ_Y apart in the Y-direction, and we also neglect gravity.

The system can be described by the following parameters: the deflector angle θ, the restitution coefficients e and e_W, the two distributions ρ_Y and ρ_Z, and two dimensionless parameters $V = \frac{\mu_Y}{a}$ and $S = \frac{\sigma_Z}{a}$ which measure the particle denseness in Y-direction and Z-direction respectively. We denote the effective mean force experienced by the oblique wall as F_{mean}. That is, F_{mean} is defined as the average impulse experienced by the wall per unit time. Then the dimensionless force $f_{mean} = \frac{F_{mean}}{mv_0^2/\mu_Y}$ represents the average dimensionless impulse on the wall per particle in the jet.

3. Theoretical approach and results

For sufficiently dilute jets, most particles experience either zero or one particle–particle collision. Then the second and higher order collisions have negligible contributions to f_{mean}. In order to investigate the first effects of collisions between particles, we assume that each particle can only experience one collision with another particle. After this particle–particle collision, one or both of the two particles can hit the wall again, but we will neglect further particle–particle collisions. Because of the randomness in the positions of the two particles in the jet, there are two possible outcomes. The first possibility is that both particles collide with the wall and propagate to infinity without any particle–particle collision. The second possibility is that a particle can collide with another one after it rebounds from the wall, after which one or both of the particles may hit the wall again only once and propagate to infinity.

Next we analyze the probability that a given particle experiences a particle–particle collision, and hence derive an expression for f_{mean}. We denote the n-th particle by B_n, and define the following events:

$$C_{n,j} = \{B_n \text{ collides with } B_j\}, \qquad D_n = \{B_n \text{ collides with any of the previous particles}\} \quad \text{for all } n, j \in \mathbb{Z}.$$

According to our assumption, B_n may collide with at most one of the particles $\{B_0, B_1, \ldots, B_{n-1}, B_{n+1}, B_{n+2}, \ldots\}$ before propagating to infinity without experiencing further collisions. That is $C_{n,n} = \emptyset$ and $C_{n,j} \cap C_{n,k} = \emptyset$ $(\forall j \neq k)$. Here we consider the simple case where B_n may only collide with its nearest neighbors B_{n-1} or B_{n+1}. That is, $C_{n,n-j} = \emptyset$ ($2 \leqslant j \leqslant n$, $\forall n, j \in \mathbb{Z}$). In this case, we have $\mathbb{P}(D_n) = \mathbb{P}(\bigcup_{j=0}^{n-1} C_{n,j}) = \mathbb{P}(C_{n,n-1})$. Note that we assume that each particle can experience at most one particle–particle collision and hence B_n may collide with B_{n-1} if and only if B_{n-1} does not collide with the previous particle B_{n-2}. Thus we can write the probability of D_n as follows,

$$\mathbb{P}(D_n) = \mathbb{P}(C_{n,n-1}) = \mathbb{P}(C_{n,n-1}|\overline{D_{n-1}}) \cdot \mathbb{P}(\overline{D_{n-1}}) = \mathbb{P}(C_{n,n-1}|\overline{D_{n-1}}) \cdot \big(1 - \mathbb{P}(D_{n-1})\big), \tag{1}$$

where overbar denotes complement. Let p denote the probability for a single particle to collide with previous particles. When a steady state has been achieved, there will be of no difference between the states of B_n and the states of other particles, and hence letting $n \to +\infty$ in (1), we can obtain $p = \lim_{n\to+\infty} \mathbb{P}(D_n) = \frac{p_\tau}{1+p_\tau}$ where $p_\tau = \mathbb{P}(C_{n,n-1}|\overline{D_{n-1}})$ ($\forall n \in \mathbb{Z}$). In principle, we can follow this procedure to include more particle–particle collisions in calculating the probability that a given particle experiences a particle–particle collision. We have shown that including interactions with more particles just gives small corrections to the formula $p = \frac{p_\tau}{1+p_\tau}$ for a dilute system [1]. So for simplicity we choose $p = \frac{p_\tau}{1+p_\tau}$ as the approximate probability for a single particle to collide with previous particles in the system.

The force experienced by the oblique wall consists of two components. One, denoted by F_w, is the impact caused by the direct particle–wall collision without any collision with previous particles. The other, denoted by F_{cw}, is the impact caused by the particle–wall collision after the particle–particle collision. For example, B_n can directly hit the wall if and only if it does not collide with the previous particle B_{n-1}, or B_n can hit the wall after the particle–particle collision with B_{n-1} only if B_{n-1} does not hit its previous particle B_{n-2}. Therefore we can write the total average impulse on the wall F_{mean} as

$$F_{mean} = \mathbb{E}(F_w) + \mathbb{E}(F_{cw}) = \mathbb{E}(F_w|\overline{D_n})\big(1 - \mathbb{P}(D_n)\big) + \mathbb{E}(F_{cw}|\overline{D_{n-1}})\big(1 - \mathbb{P}(D_{n-1})\big). \tag{2}$$

For a steady state and $n \to +\infty$, we obtain,

$$F_{mean} = (1-p)\Big(\lim_{n\to+\infty} \mathbb{E}(F_w|\overline{D_n}) + \lim_{n\to+\infty} \mathbb{E}(F_{cw}|\overline{D_{n-1}})\Big) = \frac{1}{1+p_\tau}\Big(F_b + \iint_A F_p \rho(z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\Big) \tag{3}$$

where F_b denotes the impact on the wall caused by a single particle to which the first collision is with the wall, and F_p denotes the impact on the wall after the particle collides with the previous neighbor when the neighbor particle itself does not experience a collision. $\rho(z_1, z_2)$ is the joint distribution density of z_1 and z_2 which denote the initial heights of two particles, and $A = \{(z_1, z_2) \in \mathbb{R}^2 \mid C_{2,1}\}$. Using (3), a straightforward calculation [1] gives the analytical formula for the dimensionless mean force f_{mean} as

$$\begin{aligned} f_{mean} = {} & \frac{(1+e_w)\sin\theta}{4 + 2\,\mathrm{erf}(C_-) - 4\,\mathrm{erf}(C_+)} \cdot \Big\{4 + \big((1+e) - e_w(1-e)\big)\big(\mathrm{erf}(D_-) - \mathrm{erf}(D_+)\big) + \big((1-e) \\ & - e_w(1+e)\big)\big[\mathrm{erf}(C_-) - \mathrm{erf}(C_+) + \mathrm{H}(\omega)\big(\mathrm{erf}(K_+) - \mathrm{erf}(K_-)\big)\big] + (1+e)(1+e_w) \\ & \times \Big[\Big(\frac{S^2\sin^2\theta}{2} + \frac{V^2\cos^2\theta}{4}\Big)\big(\mathrm{erf}(C_-) - \mathrm{erf}(C_+) + \mathrm{erf}(D_+) - \mathrm{erf}(D_-) + \mathrm{H}(\omega)\big(\mathrm{erf}(K_+) - \mathrm{erf}(K_-)\big)\big) \\ & + \frac{VS\sin\theta\cos\theta}{\sqrt{\pi}}\big(e^{-C_+^2} - e^{-C_-^2} + e^{-D_-^2} - e^{-D_+^2} + \mathrm{H}(\omega)\big(e^{-K_-^2} - e^{-K_+^2}\big)\big) \\ & + \frac{S^2\sin^2\theta}{\sqrt{\pi}}\big(C_+e^{-C_+^2} - C_-e^{-C_-^2} + D_-e^{-D_-^2} - D_+e^{-D_+^2} + \mathrm{H}(\omega)\big(K_-e^{-K_-^2} - K_+e^{-K_+^2}\big)\big)\Big]\Big\} \end{aligned} \tag{4}$$

where erf(x) and H(x) are Error function and Heaviside function respectively, and

$$\omega = e_w - \frac{1-e}{1+e}, \qquad C_\pm = \frac{\mp 1 - \frac{V\cos\theta}{2}}{S\sin\theta}, \qquad D_\pm = \frac{\mp\sqrt{\frac{(1+e) - e_w(1-e)}{(1+e)(1+e_w)}} - \frac{V\cos\theta}{2}}{S\sin\theta},$$

$$K_\pm = \frac{\mp\sqrt{\frac{e_w(1+e) - (1-e)}{(1+e)(1+e_w)}} - \frac{V\cos\theta}{2}}{S\sin\theta}.$$

One may naively imagine that larger θ implies a larger velocity component perpendicular to the wall, and hence larger force. We refer to this as a geometric effect. However, one can readily show that f_{mean} can be a non-monotonic function of θ. This occurs because particles that rebound from the wall can collide with incoming particles and be scattered. We refer to this as a shielding effect. Our theory allows us to quantify theses two effects, and it is the competition between them determines the behavior of f_{mean} [1].

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