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Strong asymptotics for Bergman polynomials over non-smooth domains

Estimations asymptotiques fortes pour les polynômes de Bergman sur des domaines ayant une frontière analytique par morceaux

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ABSTRACT

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\Gamma := \partial G$ is a Jordan curve, and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G. This is defined as the sequence

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \; n = 0, 1, 2, \ldots,$$

of polynomials that are orthonormal with respect to the inner product $\langle f, g \rangle := \int_G f(z)\overline{g(z)}\,\mathrm{d}A(z)$, where $\mathrm{d}A$ stands for the area measure. The aim of this Note is to report on results regarding the strong asymptotics of p_n and λ_n, $n \in \mathbb{N}$, under the assumption that Γ is piecewise analytic. These results complement an investigation started in 1923 by T. Carleman, who derived the strong asymptotics for domains with analytic boundaries and carried over by P.K. Suetin in the 1960's, who established them for domains with smooth boundaries.

RÉSUMÉ

Soit G un domaine simplement connexe dans le plan complexe $\mathbb{C}$, avec une frontière $\Gamma := \partial G$ qui est une courbe de Jordan, et soit $\{p_n\}_{n=0}^{\infty}$ les polynômes de Bergman associés a G. Plus precisémént la suite

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \; n = 0, 1, 2, \ldots,$$

des polynômes de Bergman est orthonormal pour le produit scalaire $\langle f, g \rangle := \int_G f(z)\overline{g(z)}\,\mathrm{d}A(z)$, ou $\mathrm{d}A$ est la mesure de surface. On obtient des estimations asymptotiques fortes pour p_n et λ_n, $n \in \mathbb{N}$, sous l'hypothèse que Γ est analytique par morceaux. Le resultat obtenu complète une étude commencés par T. Carleman en 1923, pour des domaines avec une frontière analytique, et continué par P.K. Suetin dans les années 1960, pour des domaines avec une frontiere régulière.

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Table 1
The rate of decay of α_n for the unit half-disk.

n	α_n	s
51	0.003 263 458 678	–
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

1. Introduction and main results

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\Gamma := \partial G$ is a Jordan curve and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G. This is defined as the sequence of polynomials

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0,\ n = 0, 1, 2, \ldots,$$

that are orthonormal with respect to the inner product $\langle f, g\rangle := \int_G f(z)\overline{g(z)}\,\mathrm{d}A(z)$, where $\mathrm{d}A$ stands for the area measure. Let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the complement of $\overline{G}$ and let Φ denote the conformal map $\Omega \to \Delta := \{w\colon\ |w| > 1\}$, normalized so that near infinity

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots, \quad \gamma > 0,$$

where $1/\gamma$ gives the (*logarithmic*) *capacity* $\operatorname{cap}(\Gamma)$ of Γ.

The main purpose of this Note is to report on the strong asymptotics of the leading coefficients $\{\lambda_n\}_{n\in\mathbb{N}}$ and the Bergman polynomials $\{p_n\}_{n\in\mathbb{N}}$, in cases when the boundary Γ of Ω is non-smooth. More precisely, we assume that Γ is piecewise analytic without cusps, i.e., without zero or 2π angles. Thus, we allow Γ to have corners. In this sense, our results complement an investigation started by T. Carleman [1] in 1923, who derived the strong asymptotics under the assumption that Γ is analytic, and was carried over by P.K. Suetin [8] in the 1960's, who verified them for smooth Γ. We also note [5] for a recent refinement of Carleman's theorem. Our main results are the following two theorems:

Theorem 1.1. *Assume that the boundary Γ of G is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$,*

$$\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n, \quad \text{where } 0 \leqslant \alpha_n \leqslant c_1(\Gamma)\frac{1}{n}. \tag{1}$$

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, for any $n \in \mathbb{N}$ and $z \in \Omega$,*

$$p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\bigl\{1 + A_n(z)\bigr\}, \quad \text{where } \bigl|A_n(z)\bigr| \leqslant \frac{c_2(\Gamma)}{\operatorname{dist}(z,\Gamma)|\Phi'(z)|}\frac{1}{\sqrt{n}} + c_2(\Gamma)\frac{1}{n}. \tag{2}$$

Above and in the sequel we use $c_1(\Gamma)$, $c_2(\Gamma)$, etc., to denote non-negative constants that depend only on Γ. We also use $\operatorname{dist}(z,\Gamma)$ to denote the (Euclidean) distance of z from Γ.

Regarding the sharpness of the estimate of α_n in (1), we consider the case where G is the unit half-disk and display in Table 1 a range of computed values of α_n, for $n = 51, \ldots, 60$. These values were obtained after constructing (in finite precision) the Bergman polynomials by means of the Gram–Schmidt process, and using the exact value of γ. In view of Theorem 1.1 we test the hypothesis $\alpha_n \approx 1/n^s$. The reported values of s indicate clearly that $\alpha_n \approx 1/n$. Exactly the same behaviour was observed in a number of different non-smooth cases, with various angles involved.

2. Applications

Strong asymptotics for orthogonal polynomials with respect to measures supported on the real line play a central role in the development of the theory of orthogonal polynomials in $\mathbb{R}$. This we expect to be the case for Bergman polynomials also. Accordingly, we show how Theorems 1.1 and 1.2 can be used in order to refine a classical and some recent results in the theory of complex orthogonal polynomials.

2.1. Zeros of the Bergman polynomials

When G is a bounded Jordan domain, a well-known result of Fejèr asserts that *all the zeros of* $\{p_n\}_{n\in\mathbb{N}}$, *are contained on the convex hull* $\mathrm{Co}(\overline{G})$ of $\overline{G}$. This was refined by Saff [7] to the two-dimensional interior of $\mathrm{Co}(\overline{G})$. In the above it should be added a result of Widom [9] to the effect that, *on any closed subset B of $\Omega \cap \mathrm{Co}(\overline{G})$ and for any $n \in \mathbb{N}$, the number of zeros of p_n on B is bounded independently of n*. This of course, doesn't preclude the possibility that, if $B \neq \emptyset$, p_n has a zero on B, for every $n \in \mathbb{N}$. Our result, which is a simple consequence of Theorem 1.2, shows that, under an additional assumption on $\Gamma = \partial G$, the zeros of the sequence $\{p_n\}_{n\in\mathbb{N}}$ cannot be accumulated in Ω.

Theorem 2.1. *Assume that Γ is piecewise analytic without cusps. Then, for any closed set $B \subset \Omega$, there exists $n_0 \in \mathbb{N}$, such that for $n \geqslant n_0$, p_n has no zeros on B.*

2.2. Ratio asymptotics

We derive the ratio asymptotics as a straightforward consequence of Theorems 1.1 and 1.2. Thus, we are obliged to assume that Γ is piecewise analytic without cusps. However, we believe that the result of the following corollary remains valid under much weaker assumptions on Γ.

Corollary 2.2. *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$,*

$$\sqrt{\frac{n+1}{n+2}}\,\frac{\lambda_{n+1}}{\lambda_n} = \gamma + \beta_n, \quad \text{where } |\beta_n| \leqslant c_3(\Gamma)\frac{1}{n}. \tag{3}$$

Also, for any $z \in \Omega$, and sufficiently large n,

$$\sqrt{\frac{n+1}{n+2}}\,\frac{p_{n+1}(z)}{p_n(z)} = \Phi(z)\bigl\{1 + B_n(z)\bigr\}, \quad \text{where } \bigl|B_n(z)\bigr| \leqslant \frac{c_4(\Gamma)}{\mathrm{dist}(z,\Gamma)|\Phi'(z)|}\frac{1}{\sqrt{n}} + c_4(\Gamma)\frac{1}{n}. \tag{4}$$

Since $\mathrm{cap}(\Gamma) = 1/\gamma$, (3) provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials. Furthermore, (4) suggests a simple numerical method for computing approximations to the conformal mapping $\Phi : \Omega \to \Delta$. This is quite appealing, in the sense that the Bergman polynomials alone suffice to provide approximations to both interior (via the well-known Bergman kernel method) and exterior conformal mapping associated with the same Jordan curve.

2.3. Finite recurrence relations and Dirichlet problems

Definition 2.3. We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $(N+1)$-*term recurrence relation*, if for any $n \geqslant N-1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \cdots + a_{n-N+1,n}p_{n-N+1}(z).$$

An application of the ratio asymptotics for $\{p_n\}_{n\in\mathbb{N}}$, given in Corollary 2.2, leads to the next two theorems. These refine, respectively, Theorems 2.2 and 2.1 of [4], in the sense that they weaken the C^2-smoothness assumption on Γ. For their proof, it is sufficient to note that: (a) the two theorems are equivalent to each other and (b) the reason for assuming that Γ is C^2-smooth in Theorem 2.2 of [4] was to ensure the ratio asymptotics of the Bergman polynomials; see [4, §4 Rem. (i)].

Theorem 2.4. *Assume that Γ is piecewise analytic without cusps. If the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $(N+1)$-term recurrence relation, with some $N \geqslant 2$, then $N = 2$ and Γ is an ellipse.*

Theorem 2.5. *Let G be a bounded simply-connected domain with Jordan boundary Γ, which is piecewise analytic without cusps. Assume that there exists a positive integer $N := N(G)$ with the property that the Dirichlet problem*

$$\Delta u = 0 \quad \text{in } G, \qquad u = \bar{z}^m z^n \quad \text{on } \Gamma, \tag{5}$$

has a polynomial solution of degree $\leqslant m(N-1)+n$ in z and of degree $\leqslant n(N-1)+m$ in $\bar{z}$, for all positive integers m and n. Then Γ is an ellipse and $N = 2$.

Theorem 2.5 confirms a special case of the so-called Khavinson and Shapiro conjecture; see e.g. [3]. We note that the equivalence between the two properties "the Bergman polynomials of G satisfy a finite-term recurrence relation" and "any Dirichlet problem in G, with polynomial data, possesses a polynomial solution" was first established in [6].

We conclude by noting that Theorems 1.1 and 1.2 can be used in order to relax the smoothness boundary conditions in [2, §4], regarding the asymptotics of orthogonal polynomials defined on an archipelago.

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