C. R. Acad. Sci. Paris, Ser. I 348 (2010) 71–73

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Analytic Geometry

A Note on the cone of mobile curves

Une Note sur le cône des courbes mobiles

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ARTICLE INFO

Article history:
Received 4 September 2009
Accepted after revision 10 November 2009
Available online 24 December 2009

Presented by Jean-Pierre Demailly

ABSTRACT

S. Boucksom, J.-P. Demailly, M. Păun and Th. Peternell proved that the cone of mobile curves $\overline{ME(X)}$ of a projective complex manifold X is dual to the cone generated by classes of effective divisors and conjectured an extension of this duality in the Kähler set-up. We show that their conjecture implies that $\overline{ME(X)}$ coincides with the cone of integer classes represented by closed positive smooth $(n-1,n-1)$-forms. Without assuming the validity of the conjecture we prove that this equality of cones still holds at the level of degree functions.

RÉSUMÉ

S. Boucksom, J.-P. Demailly, M. Păun et Thomas Peternell ont montré que le cône des courbes mobiles $\overline{ME(X)}$ d'une variété projective complexe X est le dual du cône engendré par les classes de diviseurs effectifs, et ils ont conjecturé que cette dualité pouvait s'étendre dans le contexte kählerien. Nous montrons que cette conjecture implique que $\overline{ME(X)}$ coïncide avec le cône des classes entières représentées par des formes positives fermées de type $(n-1,n-1)$ et de classe C^∞. Sans supposer que cette conjecture soit vraie, nous montrons que cette égalité de cônes a lieu en tout cas au niveau des fonctions degré associées.

Let X be a smooth complex projective variety of dimension n. A curve C on X is called *mobile* if it is the member of an algebraic family of (generically) irreducible curves covering X. Let $\overline{ME(X)}$ denote the closed convex cone generated by classes of mobile curves inside $N_1(X) := (H_{\mathbb{R}}^{n-1,n-1}(X) \cap H^{2n-2}(X,\mathbb{Z})/\operatorname{Tors}) \otimes_{\mathbb{Z}} \mathbb{R}$. We shall call the elements of $\overline{ME(X)}$ *mobile classes*.

In [2] it is shown that the following cones in $N_1(X)$ coincide:

(1) the cone $\overline{ME(X)}$ of mobile curves,
(2) the cone $\mathcal{M}_{NS} := \mathcal{M} \cap N_1$, where $\mathcal{M} \subset H_{\mathbb{R}}^{n-1,n-1}(X)$ is the closure of the convex cone generated by cohomology classes of currents of the type $\nu_*(\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_{n-1})$ for Kähler forms $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}$ on a modification $\nu : \tilde{X} \to X$ on X,
(3) the dual cone $(\mathcal{E}_{NS})^\vee$ of the cone $\mathcal{E}_{NS}$ of pseudo-effective divisors on X.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.11.003

It was known from [4] that $\mathcal{E}_{NS} = \mathcal{E} \cap NS_{\mathbb{R}}$, where $\mathcal{E}$ is the cone of classes of positive closed currents of type $(1,1)$ and $NS_{\mathbb{R}}(X) := (H^{1,1}_{\mathbb{R}}(X) \cap H^2(X,\mathbb{Z})/\mathrm{Tors}) \otimes_{\mathbb{Z}} \mathbb{R}$. In [2, Conjecture 2.3] it is further conjectured that the cones $\mathcal{M}$ and $\mathcal{E}$ are dual.

In this Note we compare $\overline{ME(X)}$ to the closed convex cone $P^{n-1,n-1}$ in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by closed positive smooth $(n-1,n-1)$-forms. From the above statements it is clear that $\overline{P^{n-1,n-1}} \cap N_1(X) \subset \overline{ME(X)}$. The converse inclusion will follow from our arguments if we admit the conjecture of Boucksom, Demailly, Păun and Peternell. If not we still get an equality at the level of degree functions as follows.

Any mobile class α gives rise to a degree function

$$\deg_\alpha : \mathrm{Pic}(X) \to \mathbb{R}, \qquad \deg_\alpha L := c_1(L)\alpha$$

and further to a notion of stability of torsion-free sheaves on X which generalizes the classical case $\alpha = H^{n-1}$ for a class H of an ample divisor, cf. [3]. On the other side we consider *semi-Kähler* metrics on X, i.e. such that their associated Kähler forms ω satisfy $\mathrm{d}\omega^{n-1} = 0$. Such a metric gives likewise a degree function:

$$\deg_\omega : \mathrm{Pic}(X) \to \mathbb{R}, \qquad \deg_\omega L := c_1(L)\big[\omega^{n-1}\big].$$

Then we can state:

Theorem. *For any mobile class α in the interior of the mobile cone $\overline{ME(X)}$ there exists a semi-Kähler metric on X with associated Kähler form ω such that $\deg_\alpha = \deg_\omega$ on $\mathrm{Pic}(X)$.*

By [6, Corollary 5.3.9] this has the following consequence on the moduli space of stable vector bundles:

Corollary. *If α is a class in the interior of $\overline{ME(X)}$, then the smooth part of the moduli space of stable vector bundles with respect to α admits a natural Kähler structure.*

Note that the statement of the theorem would be easy to prove by a direct averaging process if we knew that mobile curves are given by submersive families. But in general focal points could create singularities in the direct image of the integral mean value. As the referee suggests this could be possibly overcome by picking smoothly varying densities on the curves in such a way that their global support on the family of curves avoid critical points. We will use instead a technically easier duality argument.

We start by introducing some more notation: We denote by $\mathcal{D}'^{p,q}$ the space of currents of bidegree (p,q) (and bidimension $(n-p,n-q)$) on X. Let

$$V_{\mathrm{d}} = V_{\mathrm{d}}(X) := \big\{T \in \mathcal{D}'^{1,1}_{\mathbb{R}} \bigm| \mathrm{d}T = 0\big\}/\mathrm{dd}^c\mathcal{D}'^{0,0}_{\mathbb{R}},$$
$$V_{\mathrm{dd}^c} = V_{\mathrm{dd}^c}(X) := \big\{T \in \mathcal{D}'^{1,1}_{\mathbb{R}} \bigm| \mathrm{dd}^c T = 0\big\}/\big\{\bar\partial S + \partial \bar S \bigm| S \in \mathcal{D}'^{1,0}\big\},$$

V^+_{d}, $V^+_{\mathrm{dd}^c}$ the cones generated by positive currents in V_{d} and V_{dd^c} respectively and $V^+_{\mathrm{d},NS}$, $V^+_{\mathrm{dd}^c,NS}$ their intersections with $NS_{\mathbb{R}}(X) := (H^{1,1}_{\mathbb{R}}(X) \cap H^2(X,\mathbb{Z})/\mathrm{Tors}) \otimes_{\mathbb{Z}} \mathbb{R}$.

The following lemma makes use of the duality theorem of [2] and of a result of Alessandrini and Bassanelli on pull-backs of pluriharmonic currents [1].

Lemma. *The natural map $j : V_{\mathrm{d}} \to V_{\mathrm{dd}^c}$ induces a bijection between positive cones*

$$V^+_{\mathrm{d},NS} \to V^+_{\mathrm{dd}^c,NS}.$$

Proof. It is known that in the case of compact Kähler manifolds the map j, which associates to a class of a bidegree $(1,1)$ closed current $\{T\}_{\mathrm{d}} \in V_{\mathrm{d}}$ its class $\{T\}_{\mathrm{dd}^c} \in V_{\mathrm{dd}^c}$, is well defined and bijective.

The inclusion $j(V^+_{\mathrm{d}}) \subset V^+_{\mathrm{dd}^c}$ being obvious, we consider a dd^c-closed positive current T with $\{T\}_{\mathrm{dd}^c} \in V_{\mathrm{dd}^c,NS}$. Let $\eta := j^{-1}(\{T\}_{\mathrm{dd}^c}) \in V_{\mathrm{d}}$. We shall show that $\eta \in V^+_{\mathrm{d}}$.

By the cited result of [2] it suffices to check that for any modification $\nu : \tilde X \to X$ and Kähler forms $\tilde\omega_1, \ldots, \tilde\omega_{n-1}$ on $\tilde X$ one has

$$\eta\nu_*\big([\tilde\omega_1 \wedge \cdots \wedge \tilde\omega_{n-1}]\big) \geqslant 0.$$

But

$$\eta\nu_*\big([\tilde\omega_1 \wedge \cdots \wedge \tilde\omega_{n-1}]\big) = \nu^*(\eta)[\tilde\omega_1 \wedge \cdots \wedge \tilde\omega_{n-1}] = \nu^*\big(\{T\}_{\mathrm{dd}^c}\big)[\tilde\omega_1 \wedge \cdots \wedge \tilde\omega_{n-1}]$$

and Theorem 3 of [1] asserts that $\nu^*(\{T\}_{\mathrm{dd}^c}) \in V^+_{\mathrm{dd}^c}(\tilde X)$, whence the desired inequality. $\square$

We can now give the proof of the theorem.

Proof. Let α be an element in the interior of the cone of mobile curves. We denote by $\mathcal{D}'^{1,1}_{+}$ the cone of positive currents inside the space $\mathcal{D}'^{1,1}$ of bidegree $(1,1)$ currents on X. We fix a Kähler form σ on X and set $\mathcal{D}'^{1,1}_{+,\sigma} := \{T \in \mathcal{D}'^{1,1}_{+} \mid \int_X T \wedge \sigma^{n-1} = 1\}$. This is a compact set for the weak topology on $\mathcal{D}'^{1,1}$ [5, III.1.23]. Let $\beta_1, \ldots, \beta_k \in H^{n-1,n-1}_{\mathbb{R}}(X)$ be such that $V_{\mathrm{dd}^c,NS} = \{t \in V_{\mathrm{dd}^c} \mid t\beta_1 = 0, \ldots, t\beta_k = 0\}$ and set $W := \{T \in \mathcal{D}'^{1,1} \mid \mathrm{dd}^c T = 0,\ \{T\}_{\mathrm{dd}^c}\alpha = 0,\ \{T\}_{\mathrm{dd}^c}\beta_1 = 0, \ldots, \{T\}_{\mathrm{dd}^c}\beta_k = 0\}$. Remark that W and $\mathcal{D}'^{1,1}_{+,\sigma}$ are disjoint. Indeed, if $T \in \mathcal{D}'^{1,1}_{+,\sigma}$ were dd^c-closed and $\{T\} \in V^{+}_{\mathrm{dd}^c,NS}$, then by the lemma there would exist a d-closed positive current $S \in \mathcal{D}'^{1,1}_{+}$ such that $\{T\}_{\mathrm{dd}^c} = j(\{S\}_{\mathrm{d}})$ and in particular

$$\{T\}_{\mathrm{dd}^c}\alpha = \{S\}_{\mathrm{d}}\alpha > 0.$$

The Hahn–Banach theorem then implies the existence of a functional on $\mathcal{D}'^{1,1}$, which vanishes on W and is positive on $\mathcal{D}'^{1,1}_{+,\sigma}$. This functional is thus given by a real $(n-1,n-1)$-form u on X. The form u is strictly positive on X since the functional is positive on $\mathcal{D}'^{1,1}_{+,\sigma}$. The vanishing on W implies that u is also d-closed. As functionals on $NS_{\mathbb{R}}(X)$, $[u]$ and α have the same kernel, hence they coincide up to some multiplicative constant.

It is enough now to take a positive $(n-1)$-st root ω of u. For the convenience of the reader we show how this is done. Remark first that

$$\left(i \sum_{1\leqslant i,j\leqslant n} a_{ij}\,\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j\right)^{n-1} = (n-1)!\,i^{(n-1)^2} \sum_{1\leqslant i,j\leqslant n} (-1)^{i+j} c_{ji}\,\mathrm{d}\hat{z}_i \wedge \mathrm{d}\hat{\bar{z}}_j, \tag{1}$$

where we denoted by c_{ij} the cofactor of a_{ij} in the matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ and $\mathrm{d}\hat{z}_i := \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{i-1} \wedge \mathrm{d}z_{i+1} \wedge \cdots \wedge \mathrm{d}z_n$, $\mathrm{d}\hat{\bar{z}}_j := \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_{j-1} \wedge \mathrm{d}\bar{z}_{j+1} \wedge \cdots \wedge \mathrm{d}\bar{z}_n$. The relation ${}^tCA = \det(A)I_n$ for the matrix of cofactors $C = (c_{ij})_{1\leqslant i,j\leqslant n}$ implies

$$A = \sqrt[n-1]{\det(C)}\,{}^tC^{-1}$$

when A is positive definite. When the matrix C is positive definite, one obtains a unique positive definite solution A of Eq. (1). $\square$

Acknowledgement

I wish to thank Sébastien Boucksom and the anonymous referee for their comments on this Note.

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