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Mathematical Analysis

Sharp constants in the Paneyah–Logvinenko–Sereda theorem

Sur les constantes optimales dans le théorème de Paneyah–Logvinenko–Sereda

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ABSTRACT

We shall find some sharp constants in one type of uncertainty principle — Paneyah–Logvinenko–Sereda theorem.

RÉSUMÉ

On trouve la norme de l'opérateur inverse de l'opérateur de restriction pour deux types d'ensembles dans la classe des fonctions de Paley–Wiener.

1. Introduction

Consider the complex-valued function $f \in L^2(\mathbb{R}) = L^2(\mathbb{R}, m)$, where m is the Lebesgue measure on $\mathbb{R}$. The Fourier transform $\hat{f}$ of f is defined as follows:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{i\xi t}\, dt, \quad \xi \in \mathbb{R}, \tag{1}$$

so

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{-it\xi}\, d\xi, \quad t \in \mathbb{R}, \tag{2}$$

and

$$\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}.$$

The integrals in (1) and (2) exist in the sense of Plancherel's theorem. We say that the closed support of $\hat{f}$ is the *spectrum* of f and write $\operatorname{spec}(f)$. For $\sigma > 0$ put

$$\mathfrak{E}_\sigma = \{ f \in L^2(\mathbb{R}) \colon \operatorname{spec}(f) \subset [-\sigma, \sigma] \}.$$

The class $\mathfrak{E}_\sigma$ is very important in harmonic analysis. By Paley–Wiener theorem $f \in \mathfrak{E}_\sigma$ if and only if f is an analytic function on $\mathbb{C}$ and the exponential type of f is not more than σ.

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Consider now a measurable set $S \subset \mathbb{R}$. We say that S is *essential* if for some $\sigma > 0$ there exists a constant $C(S,\sigma)$ such that the inequality

$$\int_{\mathbb{R}} \left|f(x)\right|^2 \mathrm{d}x \leqslant C(S,\sigma) \int_{S} \left|f(x)\right|^2 \mathrm{d}x \tag{3}$$

holds for every $f \in \mathfrak{E}_\sigma$. If S is essential then such a constant exists for every $\sigma > 0$. Below we shall always assume that $C(S,\sigma)$ is the sharp constant. B. Paneyah proved the theorem:

Theorem 1 *(B. Paneyah). The following two conditions are equivalent*: 1. *S is an essential set*; 2. *S is relatively dense.*

Condition 2 means that there exist constants r and $\delta > 0$ such that $m([x-r, x+r] \cap S) > \delta$ for every $x \in \mathbb{R}$.

There are many proofs of the above theorem and each of them gives some estimates on $C(S,\sigma)$ but these estimates are not sharp (see, for example, [1]). We determine sharp constants for two specific sets S: when $S = \mathbb{R} \setminus [-R, R]$ for some $R > 0$ and when $S = \bigcup_{n\in\mathbb{Z}} [nl - R, nl + R]$ for some $l > 2R > 0$. A similar question for general model spaces K_θ was considered in [4].

2. Main results

We prove the following theorems:

Theorem 2.1. *Let $R > 0$ and $S = \mathbb{R} \setminus [-R, R]$. Denote $C(R,\sigma) = C(S,\sigma)$. Then*

$$C(R,R) \sim \frac{1}{1 - \frac{2R^2}{\pi}} \sim 1 + \frac{2R^2}{\pi}, \quad \text{when } R \to 0, \tag{4}$$

$$C(R,R) = \frac{\mathrm{e}^{2R^2}}{4R\sqrt{\pi}}\left(1 + O\left(\frac{1}{R^2}\right)\right), \quad R \to \infty. \tag{5}$$

Also

$$C(R,\sigma) = C\left(\sqrt{R\sigma}, \sqrt{R\sigma}\right), \tag{6}$$

so

$$C(R,\sigma) = C\left(\sqrt{R\sigma}, \sqrt{R\sigma}\right) \sim 1 + \frac{2R\sigma}{\pi}, \quad R\sigma \to 0, \tag{7}$$

$$C(R,\sigma) = \frac{\mathrm{e}^{2R\sigma}}{4\sqrt{\pi R\sigma}}\left(1 + O\left(\frac{1}{R\sigma}\right)\right), \quad R\sigma \to \infty. \tag{8}$$

Lemma 2.2. *Let $R > 0$, $l > 2R$,*

$$S = \bigcup_{n\in\mathbb{Z}} [nl - R, nl + R].$$

Denote $C(R,l,\sigma) = C(S,\sigma)$. Then

$$C(R,l,\sigma) = C\left(\frac{2\pi R}{l}, 2\pi, \frac{l\sigma}{2\pi}\right).$$

Now to the 2π-periodic function w we associate the matrix

$$M_n(w) = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} & c_n \\ \bar{c}_1 & c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \bar{c}_n & \bar{c}_{n-1} & \bar{c}_{n-2} & \dots & \bar{c}_1 & c_0, \end{pmatrix},$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t.$$

Theorem 2.3. *Let S be as in Lemma 2.2 with $l = 2\pi$. Then for $\sigma \leqslant \frac{1}{2}$, $C(R, 2\pi, \sigma) = \frac{\pi}{R}$ and for every function $f \in \mathfrak{E}_\sigma$ we can put $=$ in (3) instead of $\leqslant$.*

Let $\frac{1}{2} < \sigma \leqslant 1$. Then

$$C(R, l, \sigma) = \frac{\pi}{\frac{2\pi R}{l} - \sin(\frac{2\pi R}{l})}.$$

In general, if $\frac{n}{2} < \sigma \leqslant \frac{n+1}{2}$ for some integer n, then $C(S, \sigma) = \lambda_n^{-1}$, where λ_n is the smallest eigenvalue of the matrix $M_n(\chi_S)$ and

$$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}$$

Remark 1. Note that if $w = \chi_S$ then

$$c_k = \frac{\sin(kR)}{R}.$$

Theorem 2.4. *Let S, σ be as in above theorem. Let $\frac{n}{2} < \sigma \leqslant \frac{n+1}{2}$ for some integer n. Let $y = (y_1, \dots y_{n+1})$ be the eigenvector of M_n such that $M_n y = \lambda_n y$. For $0 \leqslant k \leqslant n$ put $J_k = (n - k - \sigma, \sigma - k)$. Denote now*

$$u_{ex}(\xi) = \begin{cases} y_k, & \xi \in J_k \\ 0, & \xi \notin \bigcup J_k \end{cases}$$

and

$$f_{ex}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} u_{ex}(\xi) \mathrm{e}^{-\mathrm{i}\xi x} \, \mathrm{d}\xi.$$

Then

$$\int\limits_{\mathbb{R}} \left| f_{ex}(x) \right|^2 \mathrm{d}x = C(S, \sigma) \int\limits_{S} \left| f_{ex}(x) \right|^2 \mathrm{d}x.$$

Theorem 2.5. *Let w be a measurable nonnegative 2π-periodic function which is positive on a set of positive measure. Put $L^2(w) = L^2(w \cdot dm)$ and $\|f\|_w = \|f\|_{L^2(w)}$. Assume there exists a constant Q such that*

$$\left| \sum_{|n|<N} c_n \mathrm{e}^{\mathrm{i}nx} \right| < Q$$

for every $N \in \mathbb{N}$. Then

$$\|f\|_{\mathbb{R}}^2 \leqslant \lambda_n^{-1} \|f\|_w^2,$$

where λ_n is the smallest eigenvalue of the matrix $M_n(w)$.

3. Idea of proof of Theorem 2.1

One can easily deduce (6) by scaling. We show how to find $C(R, R)$. We introduce an operator $K_1 : L^2(-R, R) \to L^2(-R, R)$:

$$K_1 u(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-R}^{R} \mathrm{e}^{-\mathrm{i}xt} u(t) \, \mathrm{d}t.$$

It is easy to see that for $f \in \mathfrak{E}_R$

$$\|f\|_{\mathbb{R}}^2 \leqslant \|f - \chi_{(-R,R)} f\|_{\mathbb{R}}^2 + \|K_1\|_2^2 \|f\|_{\mathbb{R}}^2. \tag{9}$$

Here $\|f\|_{\mathbb{R}}$ is $L^2(\mathbb{R})$-norm and $\|K_1\|_2$ is an operator norm of $K_1 : L^2(-R, R) \to L^2(-R, R)$.

K_1 is compact operator with discrete spectrum and $K_1 K_1^* = K_1^* K_1$, so

$$\|K_1\|_2 = \max\bigl\{|\lambda|: \exists u \not\equiv 0:\ K_1 u = \lambda u\bigr\} = |\mu|. \tag{10}$$

It is now obvious that the inequality (9) becomes equality for certain $f \in \mathfrak{E}_R$ (take the eigenvector u_{ex} such that $K_1 u = \mu u$ and find f such that $u = \hat{f}$).

We introduce one more operator:

$$Tu(t)=\frac{\mathrm{d}}{\mathrm{d}t}\big((1-t^2)u'(t)\big),\quad t\in(-1,1),\ u\in C_0^\infty(-1,1).$$

This operator has an extension to a self-adjoint operator T with spectrum $\{-\ell(\ell+1)\colon\ \ell\in\mathbb{Z}_+\}$. Put now

$$Au(t)=Tu(t)+R^4\big(1-t^2\big)u(t),\quad t\in(-1,1).$$

The following proposition is well known (see [3]):

Proposition 3.1. *If v is an eigenfunction of the operator A then the function $u:x\mapsto v(\frac{x}{R})$, $x\in(-R,R)$, is an eigenfunction of the operator K_1.*

If m_0 is the eigenvalue of the operator A with smallest absolute value and $Av=m_0v$ then $K_1u=\mu u$ (where μ is introduced in (10)*).*

The eigenfunctions of the operator A are called *Prolate Spheroidal Wave Functions*. Now we have to find v. If $R\to0$ then it is easy to find the asymptotic of v by means of the perturbation theory (see [2]).

If $R\to\infty$ then it is harder but still possible, see [3, Ch. 1, §5].

4. Brief proof of Theorem 2.3

Observe that if $\operatorname{spec}(f)\subset[-\sigma,\sigma]$ then $\operatorname{spec}(|f|^2)\subset[-2\sigma,2\sigma]$ and $\widehat{|f|^2}(\pm2\sigma)=0$. Let $\frac{n}{2}<\sigma\leqslant\frac{n+1}{2}$. We have

$$\chi_S(x)=\sum c_n\mathrm{e}^{\mathrm{i}nx},$$

so

$$\int_S\big|f(x)\big|^2\,\mathrm{d}x=\sum c_k\sqrt{2\pi}\,\widehat{|f|^2}(k)=c_0\|f\|_2^2+\sum_{0<|k|<n+1}c_k\int u(\xi)\bar u(\xi-k)\,\mathrm{d}\xi,\tag{11}$$

where $u=\hat f$. Now the first statement of Theorem 2.3 is obvious and we shall prove the general statement. We shall introduce $n+1$ vectors in $L^2(n-\sigma,\sigma)$: for $k=0,\dots,n$ put $v_k=u(\xi-k)$, $\xi\in(n-\sigma,\sigma)$. Denote $E=\operatorname{span}\{v_k\}$ and $v=(v_0,\dots,v_n)^T$. For $k=0,\dots,n-1$ put also $w_k=u(\xi-k)$, $\xi\in(\sigma-1,n-\sigma)$, $F=\operatorname{span}\{w_k\}$, $w=(w_0,\dots,w_{n-1})^T$. Then the right-hand side of (11) is equal to $(A_nv,v)+(A_{n-1}w,w)$, where A_n acts on the vector v like multiplication of the matrix M_n by the column v. One can see that the spectrum of A_n is equal to the spectrum of M_n (and the same for A_{n-1} and M_{n-1}). So

$$(A_nv,v)+(A_{n-1}w,w)\geqslant\lambda_n\|v\|^2+\lambda_{n-1}\|w\|^2\geqslant\min(\lambda_n,\lambda_{n-1})\|u\|^2_{(-\sigma,\sigma)}=\min(\lambda_n,\lambda_{n-1})\|f\|^2_{\mathbb{R}}.$$

It is very easy to see that $\lambda_n\leqslant\lambda_{n-1}$. Combining (11) and the last inequality we obtain

$$\int_S\big|f(x)\big|^2\,\mathrm{d}x\geqslant\lambda_n\|f\|^2_{\mathbb{R}}.$$

Now it is easy to see that Theorem 2.4 holds.

The second statement of Theorem 2.3 is just a corollary of the previous result.

The last theorem can be proved in the same way.

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