C. R. Acad. Sci. Paris, Ser. I 348 (2010) 89–92

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Probability Theory

Uniqueness result for the BSDE whose generator is monotonic in y and uniformly continuous in z ☆

Un résultat d'unicité pour une équation différentielle stochastique rétrograde dont le générateur g est monotone en y et uniformément continue en z

Sheng-Jun Fan, Long Jiang

College of Sciences, China University of Mining & Technology, Xuzhou, Jiangsu 221116, PR China

ARTICLE INFO

Article history:
Received 24 January 2009
Accepted after revision 6 October 2009
Available online 22 December 2009

Presented by Paul Malliavin

ABSTRACT

In this Note, we prove that if g is continuous, monotonic and has a general growth in y, g is uniformly continuous in z, and $(g(t,0,0))_{t\in[0,T]}$ is square integrable, then for each square integrable terminal condition ξ, the one-dimensional backward stochastic differential equation (BSDE) with the generator g has a unique solution. This generalizes some corresponding (one-dimensional) results.

RÉSUMÉ

Dans cette Note on démontre que si g est continue, monotone, de croissance quelconque en y, g uniformément continue en z et $(g(t,0,0))_{t\in[0,T]}$ est de carré intégrable, alors pour toute condition finale ξ de carré intégrable, en dimension un, l'équation différentielle stochastique rétrograde (BSDE) de générateur g, a une solution unique. Ce résultat généralise des résultats connus dans le cas de la dimension un.

1. Introduction

We consider the following one-dimensional backward stochastic differential equation (BSDE for short):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad t \in [0, T], \tag{1}$$

where ξ is a square integral random variable termed the terminal condition, the random function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ is progressively measurable for each (y, z), termed the generator of the BSDE (1), and B is a d-dimensional Brownian motion. The solution $(y_\cdot, z_\cdot)$ is a pair of square integrable, adapted processes. The triple (ξ, T, g) is called the parameters of the BSDE (1).

☆ Supported by the National Natural Science Foundation of China (No. 10671205), National Basic Research Program of China (No. 2007CB814901), Youth Foundation of China University of Mining & Technology (Nos. 2006A041 and 2007A029) and Qing Lan Project.

E-mail addresses: f_s_j@126.com (S.-J. Fan), jianglong365@hotmail.com (L. Jiang).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.10.023

Such equations, in nonlinear case, were firstly introduced by [11], who established an existence and uniqueness result of a solution of the BSDE (1) under the Lipschitz assumption of the generator g. Since then, many efforts have been done in relaxing the Lipschitz hypothesis on g; see, for instance [10,9,8,1–3], etc. In particular, under the conditions that g is continuous, monotonic and has a general growth in y, g is Lipschitz continuous in z, and $(g(t,0,0))_{t\in[0,T]}$ is square integrable, [12] proved the existence and uniqueness of the solution to the BSDE (1). Furthermore, [4] proved the existence of the solution to the BSDE (1) if the above Lipschitz continuity condition is replaced with the continuity and linear growth condition. Recently, under the conditions that g does not depend on y, g is uniformly continuous in z, and $(g(t,0))_{t\in[0,T]}$ is a bounded process, [7] obtained a uniqueness result on the solution of the BSDE (1).

Enlightened by these results, this Note proves that if g is continuous, monotonic and has a general growth in y, g is uniformly continuous in z, and the process $(g(t,0,0))_{t\in[0,T]}$ is square integrable, then for each square integrable terminal condition ξ, the BSDE (1) has a unique solution, which generalizes the corresponding (one-dimensional) results in [11,12,7]. It is worth mentioning that we use a different method from that used in [7], and our result does not need the condition that $(g(t,0,0))_{t\in[0,T]}$ is a bounded process.

2. Main result

Let $(\Omega,\mathcal{F},P)$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geqslant 0}$. Fix a terminal time $T>0$, let $(\mathcal{F}_t)_{t\geqslant 0}$ be the natural σ-algebra generated by $(B_t)_{t\geqslant 0}$ and assume $\mathcal{F}_T=\mathcal{F}$. For every positive integer n, we use $|\cdot|$ to denote norm of Euclidean space $\mathbf{R}^n$. For $t\in[0,T]$, let $L^2(\Omega,\mathcal{F}_t,P)$ denote the set of all $\mathcal{F}_t$-measurable random variable ξ such that $\mathbf{E}|\xi|^2<+\infty$. Let $L^2_{\mathcal{F}}(0,T;\mathbf{R}^n)$ denote the set of $\mathcal{F}_t$-progressively measurable, $\mathbf{R}^n$-valued process $\{X_t, t\in[0,T]\}$ such that

$$\|X\|_2 \hat{=} \left(\mathbf{E}\int_0^T |X_t|^2\,\mathrm{d}t\right)^{1/2} < +\infty.$$

Now, let $\xi\in L^2(\Omega,\mathcal{F}_T,P)$ be a terminal condition, g be the $\mathcal{F}_t$-progressively measurable generator of the BSDE (1). A solution of the BSDE (1) is a pair of processes $(y_\cdot, z_\cdot)$ in $L^2_{\mathcal{F}}(0,T;\mathbf{R}^{1+d})$ which satisfies BSDE (1) and $y_\cdot$ is a continuous process. In this Note, we further assume that g satisfies some of the following assumptions:

(H1) The process $(g(t,0,0))_{t\in[0,T]}\in L^2_{\mathcal{F}}(0,T;\mathbf{R}^1)$.
(H2) $\mathrm{d}P\times\mathrm{d}t-a.s.$, $(y,z)\mapsto f(\omega,t,y,z)$ is continuous.
(H3) g is monotonic in y, i.e., there exists a constant $\mu\geqslant 0$, such that, $\mathrm{d}P\times\mathrm{d}t-a.s.$,

$$\forall y_1,y_2,z,\quad \big(g(\omega,t,y_1,z)-g(\omega,t,y_2,z)\big)(y_1-y_2)\leqslant \mu|y_1-y_2|^2.$$

(H4) g has a general growth with respect to y, i.e., $\mathrm{d}P\times\mathrm{d}t-a.s.$,

$$\forall y,\quad \big|g(\omega,t,y,0)\big|\leqslant \big|g(\omega,t,0,0)\big|+\varphi\big(|y|\big),$$

where $\varphi:\mathbf{R}_+\to\mathbf{R}_+$ is an increasing continuous function.
(H5) g is uniformly continuous in z and uniform with respect to (ω,t,y), i.e., there exists a continuous, nondecreasing function $\phi(\cdot)$ from $\mathbf{R}_+$ to itself with at most linear growth and $\phi(0)=0$ such that $\mathrm{d}P\times\mathrm{d}t-a.s.$,

$$\forall y,z_1,z_2,\quad \big|g(\omega,t,y,z_1)-g(\omega,t,y,z_2)\big|\leqslant \phi\big(|z_1-z_2|\big).$$

Here and henceforth we denote the constant of linear growth for ϕ by A, i.e., $0\leqslant\phi(x)\leqslant A(x+1)$ for all $x\in\mathbf{R}_+$ (see [5] for details).
(H5′) g is Lipschitz continuous in z and uniform with respect to (ω,t,y), i.e., there exists a constant $C\geqslant 0$ such that $\mathrm{d}P\times\mathrm{d}t-a.s.$,

$$\forall y,z_1,z_2,\quad \big|g(\omega,t,y,z_1)-g(\omega,t,y,z_2)\big|\leqslant C|z_1-z_2|.$$

Remark 1. Under the conditions of (H1)–(H4) and (H5′), [12] established the existence and uniqueness of the solution to the BSDE with the generator g. This Note aims at establishing the existence and uniqueness under the conditions of (H1)–(H5). Obviously, (H5′) can imply (H5).

In the following, we will put forward and prove our main result that if g is continuous, monotonic and has a general growth in y, g is uniformly continuous in z, and the process $(g(t,0,0))_{t\in[0,T]}$ is square integrable, then the BSDE with the generator g has a unique solution, which generalizes the corresponding (one-dimensional) results in [11,12,7]. Rigorously, we have:

Theorem 1. *Assume that g satisfies* (H1)–(H5)*. Then for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the BSDE with parameters (ξ, T, g) has a unique solution $(y_\cdot, z_\cdot)$.*

Proof. Existence: Since g satisfies (H4) and (H5), then $\mathrm{d}P \times \mathrm{d}t - a.s.$, for each $(y, z) \in \mathbf{R}^{1+d}$ we have

$$\begin{aligned}\big|g(\omega,t,y,z)\big| &\leqslant \big|g(\omega,t,y,z)-g(\omega,t,y,0)\big| + \big|g(\omega,t,y,0)\big| \\ &\leqslant \phi\big(|z|\big) + \big|g(\omega,t,0,0)\big| + \varphi\big(|y|\big) \\ &\leqslant \big|g(\omega,t,0,0)\big| + A + \varphi\big(|y|\big) + A|z|.\end{aligned}$$

Thus the existence of the solution to the BSDE with parameters (ξ, T, g) follows from Theorem 4.1 in [4].

Uniqueness: Assume that $(y_\cdot, z_\cdot)$ and $(y'_\cdot, z'_\cdot)$ be two solutions to the BSDE with parameters (ξ, T, g) in $L^2_{\mathcal{F}}(0, T; \mathbf{R}^{1+d})$. Let $\hat{y}_\cdot = y_\cdot - y'_\cdot$, $\hat{z}_\cdot = z_\cdot - z'_\cdot$ then we have

$$\hat{y}_t = \int_t^T \big[g(s, y_s, z_s) - g\big(s, y'_s, z'_s\big)\big]\,\mathrm{d}s - \int_t^T \hat{z}_s \cdot \mathrm{d}B_s, \quad t \in [0, T].$$

Using the Tanaka–Meyer formula (see [6]), one gets that for each $t \in [0, T]$,

$$\begin{aligned}|\hat{y}_t| &= \int_t^T \frac{\hat{y}_s}{|\hat{y}_s|} 1_{\hat{y}_s \neq 0}\big[g(s, y_s, z_s) - g\big(s, y'_s, z'_s\big)\big]\,\mathrm{d}s - \big(L_T^0 - L_t^0\big) - \int_t^T \frac{\hat{y}_s}{|\hat{y}_s|} 1_{\hat{y}_s \neq 0}\hat{z}_s \cdot \mathrm{d}B_s \\ &= \int_t^T \big[\mu|\hat{y}_s| + 1_{\hat{y}_s \neq 0}\phi\big(|\hat{z}_s|\big)\big]\,\mathrm{d}s + (V_T - V_t) - \int_t^T \frac{\hat{y}_s}{|\hat{y}_s|} 1_{\hat{y}_s \neq 0}\hat{z}_s \cdot \mathrm{d}B_s,\end{aligned} \tag{2}$$

where L_t^0 is the local time of $\hat{y}_t$ at 0 and

$$V_t = -\int_0^t \bigg[\big(\mu|\hat{y}_s| + 1_{\hat{y}_s \neq 0}\phi\big(|\hat{z}_s|\big)\big) - \frac{\hat{y}_s}{|\hat{y}_s|} 1_{\hat{y}_s \neq 0}\big(g(s, y_s, z_s) - g\big(s, y'_s, z'_s\big)\big)\bigg]\,\mathrm{d}s - L_t^0.$$

Thanks to (H3) and (H5), we know that

$$\begin{aligned}\hat{y}_s\big[g(s, y_s, z_s) - g\big(s, y'_s, z'_s\big)\big] &= \hat{y}_s\big[g(s, y_s, z_s) - g\big(s, y'_s, z_s\big)\big] + \hat{y}_s\big[g\big(s, y'_s, z_s\big) - g\big(s, y'_s, z'_s\big)\big] \\ &\leqslant \mu|\hat{y}_s|^2 + |\hat{y}_s|\phi\big(|\hat{z}_s|\big).\end{aligned}$$

This inequality combining that L_t^0 is a continuous increasing process yields that $(V_t)_{t\in[0,T]}$ is a continuous decreasing process with $V_0 = 0$. Moreover, from (2) one also knows that

$$V_T = \hat{y}_0 - \int_0^T \big[\mu|\hat{y}_s| + 1_{\hat{y}_s \neq 0}\phi\big(|\hat{z}_s|\big)\big]\,\mathrm{d}s + \int_0^T \frac{\hat{y}_s}{|\hat{y}_s|} 1_{\hat{y}_s \neq 0}\hat{z}_s \cdot \mathrm{d}B_s,$$

then recalling that $\phi(\cdot)$ increases at most linearly, from Hölder inequality one has

$$\mathbf{E}\sup_{0\leqslant t\leqslant T} |V_t|^2 = \mathbf{E}|V_T|^2 \leqslant 4|\hat{y}_0|^2 + 8T\mathbf{E}\int_0^T \big[\mu^2|\hat{y}_s|^2 + \big(A|\hat{z}_s| + A\big)^2\big]\,\mathrm{d}s + 2\mathbf{E}\int_0^T |\hat{z}_s|^2\,\mathrm{d}s < +\infty. \tag{3}$$

In the following, for each $n \geqslant 1$, from [11], one knows that the following BSDE has a unique solution $(Y^n_\cdot, Z^n_\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R}^{1+d})$:

$$Y_t^n = \int_t^T \bigg[\mu Y_s^n + (n + 2A)\big|Z_s^n\big| + \phi\bigg(\frac{2A}{n + 2A}\bigg)\bigg]\,\mathrm{d}s - \int_t^T Z_s^n \cdot \mathrm{d}B_s, \quad t \in [0, T]. \tag{4}$$

Recalling that $\phi(\cdot)$ is a nondecreasing function from $\mathbf{R}_+$ to itself with at most linear growth, one can prove that for each $n \in \mathbf{N}$,

$$\phi(x) \leqslant (n + 2A)x + \phi\bigg(\frac{2A}{n + 2A}\bigg) \tag{5}$$

holds true for each $x \in \mathbf{R}_+$. In fact, if $0 \leqslant x \leqslant \frac{2A}{n+2A}$, the conclusion is obvious considering $\phi(\cdot)$ is nondecreasing. And, if $\frac{2A}{n+2A} < x < 1$, we have $(n+2A)x > 2A = A + A > Ax + A \geqslant \phi(x)$. Finally, in the case of $x \geqslant 1$, we also have $(n+2A)x > 2Ax = Ax + Ax \geqslant Ax + A \geqslant \phi(x)$. Therefore, for each $n \geqslant 1$, from (5) we have

$$g'(s) := \mu|\hat{y}_s| + 1_{\hat{y}_s \neq 0}\phi(|\hat{z}_s|) \leqslant \mu|\hat{y}_s| + 1_{\hat{y}_s \neq 0}(n+2A)|\hat{z}_s| + \phi\left(\frac{2A}{n+2A}\right)$$
$$= \mu(|\hat{y}_s|) + (n+2A)\left|\frac{\hat{y}_s}{|\hat{y}_s|}1_{\hat{y}_s \neq 0}\hat{z}_s\right| + \phi\left(\frac{2A}{n+2A}\right). \tag{6}$$

Obviously, $g'(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R}^1)$. Thus, considering inequalities (3) and (6), and the fact that $(0 - V_t)_{t\in[0,T]}$ is a continuous increasing process, by using Comparison Theorem 1.3 in [13] to compare the solution of the BSDE (2) with the one of the BSDE (4), we know that for each $n \geqslant 1$ and $t \in [0, T]$, $|\hat{y}_t| \leqslant Y^n_t$, $\mathrm{d}P - a.s.$

On the other hand, one may verify directly that for each $t \in [0, T]$,

$$Y^n_t = \frac{1}{\mu}\left(e^{\mu(T-t)} - 1\right)\phi\left(\frac{2A}{n+2A}\right) \quad \text{and} \quad Z^n_t \equiv 0.$$

Thus since $\phi(\cdot)$ is a continuous function and $\phi(0) = 0$, we have $|\hat{y}_t| \leqslant \lim_{n\to\infty} Y^n_t = 0$, $\mathrm{d}P - a.s.$

Hence, for each $t \in [0, T]$ we have, $y_t = y'_t$, $\mathrm{d}P - a.s.$ That is to say, the solution to the BSDE with parameters (ξ, T, g) is unique. The proof of Theorem 1 is complete. □

Remark 2. From the proof of Theorem 1, one can see that we need only the monotonicity condition in y (see (H3)) and uniform continuity condition in z (see (H5)) to ensure the uniqueness of the solution of the BSDE:

From Theorem 1 one can easily obtain the following Corollaries which can be regarded as the extensions of the corresponding (one-dimensional) results in [7,11].

Corollary 1. *Assume that g satisfies* (H1) *and* (H5)*. Moreover, let g be independent of y. Then for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the BSDE with parameters (ξ, T, g) has a unique solution.*

Corollary 2. *Assume that g satisfies* (H1) *and* (H5)*. Moreover, let g be Lipschitz continuous in y. Then for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the BSDE with parameters (ξ, T, g) has a unique solution.*

Acknowledgement

The authors would like to thank the referee for his/her careful reading and helpful suggestions.

References

[1] K. Bahlali, Backward stochastic differential equations with locally Lipschitz coefficient, C. R. Acad. Sci. Paris, Ser. I 331 (2001) 481–486.
[2] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux, L. Stoica, L^p solutions of backward stochastic differential equations, Stochastic Processes and Their Applications 108 (2003) 109–129.
[3] Ph. Briand, Y. Hu, Quadratic BSDEs with convex generators and unbounded terminal conditions, Probability Theory and Related Fields 141 (2008) 543–567.
[4] Ph. Briand, J.P. Lepetier, J. San Martin, One-dimensional BSDE's whose coefficient is monotonic in y and non-Lipschitz in z, Bernoulli 13 (1) (2007) 80–91.
[5] M.G. Crandall, Viscosity solutions — a primer, in: I. Capuzzo Dolcetta, P.L. Lions (Eds.), Viscosity Solutions and Applications, in: Lecture Notes in Mathematics, vol. 1660, Springer, Berlin, 1997, pp. 1–43.
[6] S. He, J. Wang, J. Yan, Semimartingal Theory and Stochastic Calculus, Science Press and CRC Press, Inc., 1992.
[7] G. Jia, A uniqueness theorem for the solution of backward stochastic differential equations, C. R. Acad. Sci. Paris, Ser. I 346 (2008) 439–444.
[8] M. Kobylanski, Backward stochastic differential equations and partial equations with quadratic growth, Ann. Probab. 28 (2000) 259–276.
[9] J.P. Lepetier, J. San Martin, Backward stochastic differential equations with continuous coefficient, Statistics and Probability Letters 32 (1997) 425–430.
[10] X. Mao, Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients, Stochastic Process and Their Applications 58 (1995) 281–292.
[11] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Letters 14 (1990) 55–61.
[12] E. Pardoux, BSDEs, weak convergence and homogenization of semilinear PDEs, in: Nonlinear Analysis, Differential Equations and Control, Montreal, QC, 1998, Kluwer Academic Publishers, Dordrecht, 1999, pp. 503–549.
[13] S. Peng, Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob–Meyer's type, Probability Theory and Related Fields 113 (1999) 473–499.