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Liouville and geodesic Ricci solitons

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Abstract

On a tangent bundle endowed with a pseudo-Riemannian metric of complete lift type two classes of Ricci solitons are obtained: a 1-parameter family of shrinking Liouville Ricci solitons if the base manifold is Ricci flat and a steady geodesic Ricci soliton if the base manifold is flat. A nonexistence result of geodesic Ricci solitons for the tangent bundle of a non-flat space form is also provided. ***To cite this article: M. Crasmareanu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Solitons de Ricci de type Liouville et de type géodésique. Pour le fibré tangent d'une variété équipée d'une métrique pseudo-Riemannienne ayant un relèvement complet, deux classes de solitons de Ricci sont décrits : une famille à 1 paramètre de solitons de Ricci de type Liouville contractants si la variété de base est Ricci plate, et un soliton de Ricci de type géodésique nul si celle-ci est plate. Un résultat de non-existence de solitons de Ricci géodésiques est également obtenu dans le cas du fibré tangent d'une variété non plate. ***Pour citer cet article : M. Crasmareanu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Fix M a smooth $n(>1)$-dimensional manifold. A *Ricci soliton* on M is a triple (g, V, λ) consisting in a Riemannian metric, a vector field and a real scalar satisfying the Ricci equation:

$$L_V g + 2S_g + 2\lambda g = 0$$

where S_g is the associated Ricci tensor of g and L_V is the Lie derivative with respect to V. The Ricci soliton is said to be *shrinking, steady* or *expanding* according as λ is negative, zero or positive respectively. With V a Killing vector field it results that Ricci solitons are generalizations of Einstein metrics; also for Ricci flat metrics ($S_g = 0$) it results that V is an homothetic Killing vector field. Compact Ricci solitons are fixed points of the *Ricci flow*, a very effective tool for studying the topology of manifolds [3].

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The aim of this Note is to obtain Ricci solitons in tangent bundles of Riemannian manifolds having as vector field exactly:

(1) the Liouville vector field Γ which is a global vector field on any tangent bundle, independent of any metric on the base manifold,
(2) the geodesic spray Γ_g which depends on the Riemannian metric of the base. Sometimes, Γ_g is called the *transversal Liouville vector field* [1, p. 231], or the *horizontal Liouville vector field* [5, p. 272], but we prefer this geometrical name.

Therefore we call *Liouville* and *geodesic* respectively, these types of Ricci solitons and let us remark that these classes are disjoint since Γ is a vertical vector field on *TM* while Γ_g belongs to a complementary distribution, called horizontal. We consider tangent bundles since already several studies were dedicated to the compact case.

On the tangent bundle of a Riemannian manifold there are lots of very interesting metrics [4,10], but we restrict to the pseudo-Riemannian metric of [6] and [7]; see also [8] and [9]. A strong motivation of this choice is the fact that the paper [6] contains a computation of the Ricci tensor of this metric, useful for our study. Another argument is that the present paper is dedicated to the memory of N. Papaghiuc, 1947–2008.

2. The tangent bundle with a complete lift

Fix a Riemannian metric g on the manifold M. A local system of coordinates $(x) = (x^i) = (x^1, \dots, x^n)$ on M yields a system of coordinates $(x, y) = (x^i, y^i)$ on the tangent bundle *TM*. The Levi-Civita connection (Γ^i_{jk}) of g defines a splitting $T(TM) = VTM \oplus HTM$ into vertical and horizontal vectors respectively. The integrable distribution *VTM* has the basis $(\frac{\partial}{\partial y^i})$ while the horizontal distribution *HTM* is spanned by $(\frac{\delta}{\delta x^i})$ where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^j_{i0}\frac{\partial}{\partial y^j}$, $\Gamma^j_{i0} = \Gamma^j_{ih} y^h$. Throughout the paper the transvecting with y^i will be denoted by a zero.

Consider now the kinetic energy $t(x, y) = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{ij}(x)y^i y^j$ and also the smooth functions $u, v : [0, \infty) \to \mathbb{R}$ such that [6, p. 229] $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for every t. The above conditions assure that the symmetric $(0, 2)$-type tensor field of *TM*, $G_{ij} = u(t)g_{ij} + v(t)g_{0i}g_{0j}$ is positive definite. Then, the pseudo-Riemannian metric defined in [6, p. 229] is:

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \qquad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}\right) = G_{ij}. \tag{1}$$

The tangent bundle carries a remarkable global vector field Γ, called Liouville, which is independent of any Riemannian metric on the base manifold, namely, $\Gamma = y^i \frac{\partial}{\partial y^i}$. Also, the Riemannian metric g yields the geodesic spray $\Gamma_g = y^a \frac{\delta}{\delta x^a}$. Both Γ and Γ_g are null vector fields for G. For the Lie derivative from (Ricci) we need the Lie brackets of these vector fields with the local frame fields $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$:

$$\begin{aligned} &\left[\Gamma, \frac{\delta}{\delta x^i}\right] = 0, \qquad \left[\Gamma, \frac{\partial}{\partial y^i}\right] = -\frac{\partial}{\partial y^i}, \\ &\left[\Gamma_g, \frac{\delta}{\delta x^i}\right] = \Gamma^j_{i0}\frac{\delta}{\delta x^j} + R^b_{0i0}\frac{\partial}{\partial y^b}, \qquad \left[\Gamma_g, \frac{\partial}{\partial y^i}\right] = -\frac{\delta}{\delta x^i} + \Gamma^b_{i0}\frac{\partial}{\partial y^b}. \end{aligned} \tag{2}$$

Also $\Gamma(G_{ij}) = 2t(u'g_{ij} + v'g_{i0}g_{j0}) + 2vg_{i0}g_{j0}$ and $\Gamma_g(G_{ij}) = \Gamma^b_{i0}G_{bj} + \Gamma^b_{j0}G_{ib}$.

Beginning with the Liouville vector field we have three cases:

(I) (Ricci) on $(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ yields $S_G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0$,

(II) (Ricci) on $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$ yields $S_G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 0$.

The Ricci tensor S_G of this metric is computed at the page 231 of [6] and exactly the above components are the only non-vanishing. Therefore the pair (G, Γ) belongs to a Ricci soliton if and only if G is Ricci flat which, according to Theorem 2 of the cited paper, is equivalent with the fact that (M, g) is Ricci flat and $v = u'$.

The last case, (III), (Ricci) on $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})$ yields $\Gamma(G_{ij}) + (2\lambda + 1)G_{ij} = 0$ which gives $2t(u'g_{ij} + u''g_{i0}g_{j0}) + 2u'g_{i0}g_{j0} = -(2\lambda + 1)(ug_{ij} + u'g_{i0}g_{j0})$. Applying Lemma 1 of [5] we derive the same equation $2tu' = -(2\lambda + 1)u$

with the solution $u(t) = t^{-(\lambda+\frac{1}{2})}$ which satisfies the condition $u(t) > 0$. The condition $u + 2tu' > 0$ is equivalent with $\lambda < 0$.

Theorem 2.1. *If the Riemannian manifold (M, g) is Ricci flat then the tangent bundle carries a 1-parametric family of shrinking Ricci solitons $(G_\lambda, \Gamma, \lambda)$ for $\lambda < 0$ where $G_\lambda(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G_\lambda(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0$ and:*

$$G_\lambda\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G_\lambda\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}\right) = t^{-(\lambda+\frac{3}{2})}\left[tg_{ij} - \left(\lambda + \frac{1}{2}\right)g_{i0}g_{j0})\right]. \tag{3}$$

(TM, G_λ) is also Ricci flat.

Several and very interesting examples of Ricci flat metrics are given in [2]. From Proposition 1 of the cited paper we obtain that for the pseudo-Riemannian metric G_λ the vector fields $(\frac{\delta}{\delta x^i})$ are covariant constant (parallel) with respect to vector fields $(\frac{\partial}{\partial y^i})$ and $\nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij} \frac{\partial}{\partial y^h}$.

For the geodesic vector field the same three cases occur:

(I) (Ricci) on $(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ yields: $G_{ij} + S_G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0$.
(II) (Ricci) on $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$ gives: $2S_G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = R^b_{0i0}G_{bj} + R^b_{0j0}G_{bi}$.
(III) (Ricci) on $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})$ yields: $\Gamma_g(G_{ij}) + 2\lambda G_{ij} - \Gamma^b_{i0}G_{bj} - \Gamma^b_{j0}G_{ib} = 0$.

This equation reduces to $\lambda G_{ij} = 0$ which gives $\lambda = 0$.

Using the expression of the Ricci tensor from [6, p. 231] the first equation means:

$$u - \frac{(n-1)(u'-v)}{u+2tv} = 0, \qquad v + \frac{(1-n)\alpha}{2u^2(u+2tv)} = 0 \tag{4}$$

with α from the cited paper while the second equation reduces to $2R_{ij} = uR_{0i0j} + v(R_{0j00}g_{i0} + R_{0i00}g_{j0})$.

In the following we assume that the base metric g has constant sectional curvature c. With $R_{ij} = c(n-1)g_{ij}$, $R_{0i0j} = c(2tg_{ij} - g_{i0}g_{j0})$ we get $c[2(tu - n + 1)g_{ij} - ug_{i0}g_{j0}] = 0$.

Theorem 2.2. *Let (M, g) be a flat manifold and u, v smooth functions satisfying the differential system (4). Then, on that tangent bundle (TM, G) we have a steady geodesic Ricci soliton.*

If $M_n(c)$ is a space form with $n > 1$ and $c \neq 0$ then on TM there are no geodesic Ricci solitons having G as element.

The system (4) is equivalent with:

$$2u(u+2tv)u'' - 3u(u')^2 - 2u^2v' + 5uv^2 - 2uu'v - 4tuu'v' - 2t(u')^2v + 2tv^3 = 0$$

which is divisible by $u' - v$; But $v = u'$ is not a solution of our system since from (4_1) it results $u = 0$. From Corollary 8 of [9, p. 284] it results that, although (M, g) is flat, the tangent bundle (TM, G) is not.

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