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Harmonic Analysis

Norm inequalities for convolution operators

Erlan Nursultanov [a], Sergey Tikhonov [b], Nazerke Tleukhanova [c]

[a] *Kazakh Branch of Moscow State University, Munatpasova, 7, 010010 Astana, Kazakhstan*
[b] *ICREA and Centre de Recerca Matemàtica, Apartat 50, 08193 Bellaterra, Barcelona, Spain*
[c] *Gumilyov Eurasian National University, Munatpasova, 5, 010008 Astana, Kazakhstan*

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Abstract

We study norm convolution inequalities in Lebesgue and Lorentz spaces. First, we improve the well-known O'Neil's inequality for the convolution operators and prove corresponding estimate from below. Second, we obtain Young–O'Neil-type estimate in the Lorentz spaces for the limit value parameters, i.e., $\|K * f\|_{L(p,h_1)\to L(p,h_2)}$. Finally, similar estimates in the weighted Lorentz spaces are presented. ***To cite this article: E. Nursultanov et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Inégalités de normes pour les opérateurs de convolution. Nous étudions des inégalités de normes de convolutions dans les espaces de Lebesgue et de Lorentz. En premier lieu, nous améliorons l'inégalité bien connue de O'Neil sur les opérateurs de convolution et nous établissons une minoration. En second lieu, nous donnons une estimation du type de Young–O'Neil dans les espaces de Lorentz, à savoir $\|K * f\|_{L(p,h_1)\to L(p,h_2)}$. Enfin, nous présentons des estimations similaires dans les espaces de Lorentz à poids. ***Pour citer cet article : E. Nursultanov et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let (Ω,μ) be a measurable space and $L_p(\Omega,\mu)$ be the collection of all those measurable functions f satisfying $\|f\|_{L_p(\Omega,\mu)} = (\int_\Omega |f(x)|^p\,d\mu)^{1/p} < \infty$. The distribution of a measurable function f on Ω is defined by $m(\sigma,f) = \mu\{x\in\Omega\colon |f(x)|>\sigma\}$. Then $f^*(t) = \inf\{\sigma\colon m(\sigma,f)\leqslant t\}$ is the decreasing rearrangement of f.

Let $0<p<\infty$ and $0<q\leqslant\infty$. The Lorentz space $L_{p,q}(\Omega,\mu)$ is defined [3, Ch. 4] by those measurable functions f such that

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} < \infty,$$

E-mail addresses: er-nurs@yandex.ru (E. Nursultanov), tikhonov.work@gmail.com (S. Tikhonov), tleukhanova@rambler.ru (N. Tleukhanova).

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when $0 < q < \infty$, and $\|f\|_{L_{p,\infty}} = \sup_t t^{1/p} f^*(t) < \infty$, when $q = \infty$. We also define $f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\,dt$.

Let D and Ω be measurable sets in $\mathbb{R}^n$ and $K(\cdot)$ be locally integrable function on $D - \Omega = \{x - y\colon x \in D, y \in \Omega\}$. In this paper we study norm estimates for the convolution operator

$$(Af)(y) = (K * f)(y) = \int_D K(x-y) f(x)\,dx, \quad y \in \Omega \tag{1}$$

in the Lebesgue and Lorentz spaces.

In Section 2 we consider the upper and lower estimates of $\|A\|_{L_p \to L_q}$. The upper estimate sharpens the known O'Neil and Stepanov inequalities. Section 3 is devoted to the O'Neil type inequalities in the Lorentz spaces. We study the case of limit value parameters, that is, $\|A\|_{L_{p,h_1} \to L_{p,h_2}}$ for $1 \leqslant p \leqslant \infty$. Finally, in Section 4 we give norm convolution estimates in the more general Lorentz spaces. Detailed proof of these results can be found in [9–11].

2. Norm convolution inequalities in the Lebesgue spaces

The Young convolution inequality of the form $\|K * f\|_{L_q} \leqslant \|f\|_{L_p}\|K\|_{L_r}$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ plays a very important role both in Harmonic Analysis and PDE.

O'Neil [12] extended Young's inequality as follows. Let μ be the linear Lebesgue measure and $L_p(\mathbb{R}) = L_{p,p}(\mathbb{R}, dx)$. Then $(1 < p, q, r < \infty)$

$$\|A\|_{L_p(\mathbb{R},dx) \to L_q(\mathbb{R},dx)} \leqslant C\|K\|_{L_{r,\infty}(\mathbb{R},dx)}, \tag{2}$$

where A is given by (1) with $\Omega = D = \mathbb{R}$.

Another extension of Young's inequality was proved by Stepanov [13] using the Wiener amalgam space $W(L_{r,\infty}[0,1], l_{r,\infty}(\mathbb{Z}))$ (see e.g. [6]): for $1 < p < q < +\infty$ and $1/r = 1 - 1/p + 1/q$ one has

$$\|A\|_{L_p \to L_q} \leqslant C\|K\|_{W(L_{r,\infty}[0,1], l_{r,\infty}(\mathbb{Z}))}, \tag{3}$$

where $\|K\|_{W(L_{r,\infty}[0,1],\,l_{r,\infty}(\mathbb{Z}))} := \|\|\tilde{K}\|_{L_{r,\infty}[0,1]}\|_{l_{r,\infty}(\mathbb{Z})} := \sup_{n\in\mathbb{N}} n^{1/r}(\sup_{0\leqslant t\leqslant 1} t^{1/r}\tilde{K}^*(t,\cdot))_n^*$, and $\tilde{K}(x,m) := K(m+x)$, $m \in \mathbb{Z}$, $x \in [0,1]$. In [13] it was also shown that inequalities (2) and (3) are not comparable.

In this section we sharpen O'Neil and Stepanov inequalities (2) and (3) and give an estimate of $\|A\|$ from below. We will need the following definitions. Let I be an interval with $|I| = d$. Then $T_I = \{I + kd\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}$, i.e., $\mathbb{R} = \bigcup_{k\in\mathbb{Z}}(I + kd)$. We define two collections of sets $\mathfrak{L}(I) \subset \mathfrak{U}(I)$:

$$\mathfrak{L}(I) = \left\{e\colon\ e = \bigcup_{k=1}^{m}\bigl([a,b] + kd\bigr),\ [a,b] \subseteq I,\ m \in \mathbb{N}\right\}$$

and

$$\mathfrak{U}(I) = \left\{e\colon\ e = \bigcup_{k=1}^{m}\omega_k,\ m \in \mathbb{N}\right\},$$

where $\{\omega_k\}_1^m$ is any collection of compact sets of equal measure $|\omega_k| \leqslant d$ and such that each ω_k belongs to a different element of T_I.

Theorem 1. *Let $1 < p < q < \infty$. Then for $Af = K * f$ we have*

$$C_1 \sup_I \sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/p-1/q}}\left|\int_e K(x)\,dx\right| \leqslant \|A\|_{L_p\to L_q} \leqslant C_2 \inf_I \sup_{e\in\mathfrak{U}(I)} \frac{1}{|e|^{1/p-1/q}}\left|\int_e K(x)\,dx\right|, \tag{4}$$

where the constants C_1 and C_2 depend only on p and q.

For the certain regular kernels K, for instance, monotone or quasi-monotone, the upper and lower bounds in (4) coincide, that is, we get the equivalent relation for $\|A\|_{L_p\to L_q}$.

Corollary 1. *Let $1 < p < q < \infty$ and K satisfy the following condition*:

$$\left|K(x)\right| \leqslant C\left|\frac{1}{x}\int_0^x K(t)\,\mathrm{d}t\right|, \quad x \in \mathbb{R}\setminus\{0\}.$$

Then a necessary and sufficient condition for $Af = K * f$ to be bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ is

$$S := \sup_{|x|>0} \frac{1}{|x|^{1/p-1/q}}\left|\int_0^x K(y)\,\mathrm{d}y\right| < \infty.$$

Moreover, $\|A\|_{L_p\to L_q} \approx S$.

Compared with O'Neil and Stepanov's estimates, we prove in [11] that the right-hand side estimate of (4) implies both (2) and (3). It can also be shown that for the function

$$K(x) = \begin{cases} 2^{k/r}, & \text{for } x \in [-k, -k+2^{-k}],\ k \in \mathbb{N};\\ 1, & \text{for } x \in [k, k+1/k),\ k \in \mathbb{N};\\ 0, & \text{otherwise}\end{cases}$$

we have $\inf_I \sup_{e\in\mathfrak{U}(I)} \frac{1}{|e|^{1/p-1/q}} |\int_e K(x)\,\mathrm{d}x| < \infty$, $\|K\|_{L_{r,\infty}(\mathbb{R},\mathrm{d}x)} = \infty$, $\|K\|_{W(L_{r,\infty}[0,1],\ l_{r,\infty}(\mathbb{Z}))} = \infty$.

3. Young–O'Neil type inequalities in the Lorentz spaces

The Young–O'Neil inequality for the convolution $Af = K * f$ in the Lorentz spaces is given by $\|Af\|_{L_{q,h_1}} \leqslant C\|f\|_{L_{p,h_2}}\|K\|_{L_{r,h}}$, where $1 < p, q, r < \infty$, $0 < h_1, h_2, h \leqslant \infty$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, and $\frac{1}{h} = \frac{1}{h_1} - \frac{1}{h_2}$. See O'Neil [12], Hunt [7], Yap [14], and Blozinski [4].

In this section we study the boundedness of the operator A from L_{p,h_2} into L_{p,h_1}, i.e., the limiting case of the Young–O'Neil inequality ($p = q$ and $r = 1$). First we note (see [5, Theorem 2]) that if $h_1 < h_2 \leqslant \infty$ and $K \geqslant 0$, then $A : L_{p,h_2}(\mathbb{R}, \mathrm{d}x) \to L_{p,h_1}(\mathbb{R}, \mathrm{d}x)$ implies $A \equiv 0$, i.e., $K \overset{\text{a.e.}}{=} 0$. We however show that in the case when Ω is of finite measure, the same problem has a nontrivial solution.

Next two theorems provide the boundedness of the convolution A, given by (1) with the 1-periodic functions K and f and $\Omega = D = [0, 1]$, from $L_{p,h_1}([0, 1], \mathrm{d}x)$ in $L_{p,h_2}([0, 1], \mathrm{d}x)$.

Theorem 2. *Let $1 \leqslant h_1, h_2 \leqslant \infty$, and $\frac{1}{h} = \frac{1}{h_1} - \frac{1}{h_2}$. We have*

$$\|Af\|_{L_{p,h_1}[0,1]} \leqslant C\|f\|_{L_{p,h_2}[0,1]}\left\|K^{**}\right\|_{L_{1,h}[0,1]} \tag{5}$$

for $1 < p \leqslant \infty$ and

$$\left\|(Af)^{**}\right\|_{L_{1,h_1}[0,1]} \leqslant C\left\|f^{**}\right\|_{L_{1,h_2}[0,1]}\left\|K^{**}\right\|_{L_{1,h}[0,1]}, \tag{6}$$

where $\|\varphi^{**}\|_{L_{1,h}[0,1]} \equiv (\int_0^1 (t\varphi^{**}(t))^h \frac{\mathrm{d}t}{t})^{1/h}$.

Remark. In both inequalities (5) and (6), the factor $\|K^{**}\|_{L_{1,h}}$ cannot be changed to $\|K\|_{L_{1,h}}$.

Our second goal is to give an analogue of Young–O'Neil's inequality in the $L_{\infty,q}[0, 1]$ spaces. Following Bennett et al. [2] (see also [1]), we define $L_{\infty,q}[0, 1]$ as follows

$$L_{\infty,q}[0,1] = \left\{f \in L_1[0,1]\colon\ \|f\|_{L_{\infty,q}[0,1]} := \|f\|_{L_1[0,1]} + \left(\int_0^1 \frac{(f^{**} - f^*)^q}{t}\,\mathrm{d}t\right)^{1/q} < \infty\right\}.$$

Note that $L_\infty[0, 1] = L_{\infty,1}[0, 1] \hookrightarrow L_{\infty,q}[0, 1] \hookrightarrow L_{\infty,q_1}[0, 1] \hookrightarrow L_p[0, 1]$, for $1 \leqslant p < \infty$ and $1 \leqslant q < q_1 \leqslant \infty$.

Theorem 3. *Let $1 \leqslant h_1, h_2, h < \infty$ and $\frac{1}{h} = \frac{1}{h_1} - \frac{1}{h_2}$. We have* $\|Af\|_{L_{\infty,h_1}[0,1]} \leqslant 2\|f\|_{L_{\infty,h_2}[0,1]}\|K^{**}\|_{L_{1,h}[0,1]}$.

4. Convolution operator in the Lorentz spaces with weights

In the case of non-homogeneous measures, the convolution operator does not satisfy all requirements from [12] and needs thorough investigation. The following theorem provides sufficient conditions for the convolution operator to be bounded in the weighted Lorentz space.

Theorem 4. *Let $1 < p, q < \infty$ and let the measures μ and ν be defined on measurable subsets Ω and D of $\mathbb{R}^n$, respectively. Assume that a function $K(z)$ defined on $D - \Omega = \{z = x - y\colon\ x \in D, y \in \Omega\}$ satisfies the following condition: there exists $\gamma > 0$ such that*

$$\sup_{e\in M_1} \frac{1}{(\mu(e))^{1/q'-1/\gamma p'}} \left| \int\limits_e K(x-y)\,\mathrm{d}\mu_y \right| \leqslant B \quad \text{for a.e. } x \in D,$$

$$\sup_{w\in M_2} \frac{1}{(\nu(w))^{1/p-\gamma/q}} \left| \int\limits_w K(x-y)\,\mathrm{d}\nu_x \right| \leqslant B \quad \text{for a.e. } y \in \Omega,$$

where $M_1 = \{e \subset \Omega\colon\ 0 < \mu(e) < \infty\}$ and $M_2 = \{w \subset D\colon\ 0 < \nu(w) < \infty\}$. Then

$$Af(y) = \int\limits_D K(x-y) f(x)\,\mathrm{d}\nu_x \tag{7}$$

is bounded from $L_{p,h_1}(D,\nu)$ to $L_{q,h_2}(\Omega,\mu)$ with $1 \leqslant h_1 \leqslant h_2 \leqslant \infty$ and, moreover, $\|A\|_{L_{p,h_1}(D,\nu)\to L_{q,h_2}(\Omega,\mu)} \leqslant CB$, where $C = C(p,q,h_1,h_2)$.

For the power weights the Young–O'Neil inequality was generalized by Kerman [8]. We continue this investigation by presenting the following result.

Theorem 5. *Let $\alpha, \beta \in [0,1)$, $1 < p, q < \infty$, and $0 < \frac{1}{r} = 1 - \frac{1-\alpha}{p'} - \frac{1-\beta}{q}$. Suppose that measures μ and ν are define as follows $\mu(e) = \int_e \frac{\mathrm{d}y}{|y|^\beta}$ and $\nu(\omega) = \int_\omega \frac{\mathrm{d}x}{|x|^\alpha}$. Then the convolution operator (7) with $D = \mathbb{R}$ satisfies $\|A\|_{L_p(\mathbf{R},\nu)\to L_q(\mathbf{R},\mu)} \leqslant C \sup_{0<|e|<\infty} \frac{1}{|e|^{\frac{1}{r}}} |\int_e K(t)\,\mathrm{d}t|$, where $|e|$ is the linear measure of e. Moreover, if the kernel $K(t)$ is non-negative, then $C \sup_{d>0} \frac{1}{d^{\frac{1}{r}}} \int_{-d}^{d} K(t)\,\mathrm{d}t \leqslant \|A\|_{L_p(\mathbf{R},\nu)\to L_q(\mathbf{R},\mu)}$.*

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