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ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1265–1270
COMPTES RENDUS MATHEMATIQUE

Harmonic Analysis/Calculus of Variations

Fractional integrals and A_p-weights: A sharp estimate

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Received and accepted 12 July 2009

Presented by Haïm Brezis

Abstract

We are concerned with an inequality, with an A_p weight, for Riesz potentials in $\mathbb{R}^n$. The constant in the relevant inequality is known to depend on the A_p constant of the weight. We find the exact form of this dependence. In particular, we exhibit the optimal exponent for the A_p constant of the weight. ***To cite this article: T. Alberico et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Intégrales fractionnaires et A_p-poids : une estimation optimale. On considère une inégalité à poids A_p, pour des potentiels de Riesz dans $\mathbb{R}^n$. La constante de l'inégalité dépend de la constante A_p du poids. On donne la forme exacte de la dépendance, en particulier on précise l'exposant optimal de la constante A_p du poids. ***Pour citer cet article : T. Alberico et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Des propriétés de majoration de divers opérateurs classiques en analyse harmonique dans des espaces de Lebesgue à poids peuvent être caractérisées en termes d'appartenance des poids aux classes notées A_p. Rappelons qu'un poids w, c'est-à-dire une fonction localement intégrable non négative de $\mathbb{R}^n$, appartient à la classe A_p, pour $p \in (1, \infty)$ si,

$$A_p(w) = \sup_B \left(\frac{1}{|B|} \int_B w(x)\,\mathrm{d}x \right) \left(\frac{1}{|B|} \int_B w(x)^{\frac{1}{1-p}}\,\mathrm{d}x \right)^{p-1} < \infty, \tag{1}$$

est finie, où B est une boule de $\mathbb{R}^n$, $|B|$ est la mesure de Lebesgue de B ; $A_p(w)$ est la constante A_p de w.

Introduit dans [9], on montre que l'opérateur maximal de Hardy–Littlewood est borné dans l'espace de Lebesgue $L^p(\mathbb{R}^n, w)$, muni de la mesure $w(x)\,\mathrm{d}x$, si et seulement si $w \in A_p$; on a montré, par la suite, que la condition A_p permet de décrire la classe des espaces de Lebesgue à poids, espaces dans lesquels la transformation de Hilbert, et plus généralement des opérateurs intégro-singuliers, sont bornés. Pour les résultats les plus récents de la théorie des

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.09.001

espaces A_p, en analyse harmonique, on peut consulter les monographies [14] et [6]. On rappelle seulement que les espaces A_p jouent un rôle important dans d'autres domaines de l'analyse mathématique, notamment en théorie des équations elliptiques dégénérés, en théorie du potentiel non linéaire et en théorie des fonctions quasi-régulières.

Un problème ouvert de longue date était celui de l'estimation optimale de la croissance de la norme des opérateurs introduits ci-dessus, estimation en fonction de la constante A_p du poids. Pour l'opérateur maximal de Hardy–Littlewood, le problème a été résolu dans [2], où on montre que, pour $w \in A_p$, $p \in (1,\infty)$, alors

$$\left(\int_{\mathbb{R}^n} Mf(x)^p w(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \leqslant C A_p(w)^{\frac{1}{p-1}} \left(\int_{\mathbb{R}^n} \left|f(x)\right|^p w(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{2}$$

où $C = C(p,n)$ et f localement intégrable. Ici, la puissance $\frac{1}{p-1}$ de $A_p(w)$ est la plus petite possible, c'est donc la meilleure puisque $A_p(w) \geqslant 1$. Très récemment, des résultats analogues ont été établis pour la transformation de Hilbert dans les espaces $L^p(\mathbb{R}, w)$ [11], et pour les transformations de Riesz dans $L^p(\mathbb{R}^n, w)$ [12]. La croissance exacte dans la transformation de Beurling–Ahlfors dans les espaces à poids a été déterminé dans [13] et cette propriété a des conséquences sur la régularité des solutions de l'équation de Beltrami dans le plan [1].

Dans cette Note on considère une question voisine pour l'opérateur intégrale fractionnelle I_α, opérateur aussi appelé potentiel de Riesz d'ordre $\alpha \in (0,n)$. Le potentiel de Riesz, d'ordre α, d'une fonction $f : \mathbb{R}^n \to \mathbb{R}$ est définie par :

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \quad \text{for } x \in \mathbb{R}^n. \tag{3}$$

Un résultat classique est que I_α est bornée de $L^p(\mathbb{R}^n)$ dans $L^{\frac{np}{n-\alpha}}(\mathbb{R}^n)$ pour $1 < p < \frac{n}{\alpha}$. Les inégalités à poids pour les intégrales fractionnaires peuvent être caractérisées à partir de la condition A_p. Comme il est établi dans [10], une condition nécessaire et suffisante pour le bornage de I_α de $L^p(\mathbb{R}^n, w)$ dans $L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n, w)$ est que $w^{\frac{np}{n-\alpha p}} \in A_{1+\frac{q}{p'}}$, $p' = \frac{p}{p-1}$ (voir aussi une autre approche des inégalités à poids pour des potentiels dans le cas de capacités dans [8]).

Si $w \in A_p$, $p \in (1, \frac{n}{\alpha})$, alors il existe une puissance de p, notée $\overline{p} > p$ et dépendant de $A_p(w)$ telle que I_α est borné de $L^p(B, w)$ dans $L^{\overline{p}}(B, w)$ pour toute boule $B \subset \mathbb{R}^n$ [5]. Notre résultat donne une information quantitative optimale de cette affirmation, plus précisément on a le suivant :

Théorème 0.1. *Soient $n \geqslant 2$, $\alpha \in (0,n)$, $1 < p < \frac{n}{\alpha}$ et $w \in A_p$. Alors il existe des constantes positives, $k = k(p,n)$ et $C = C(\alpha, p, n)$ telles que si*

$$p - kA_p(w)^{\frac{1}{1-p}} < q \leqslant p \tag{4}$$

alors

$$\left(\frac{1}{\int_{B_R} w(x)\,\mathrm{d}x} \int_{B_R} \left|I_\alpha f(x)\right|^{\frac{nqp}{nq-\alpha p}} w(x)\,\mathrm{d}x\right)^{\frac{nq-\alpha p}{nqp}} \leqslant C R^\alpha A_p(w)^{\frac{nq-\alpha}{nq(p-1)}} \left(\frac{1}{\int_{B_R} w(x)\,\mathrm{d}x} \int_{B_R} \left|f(x)\right|^p w(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{5}$$

pour toute boule $B_R \subset \mathbb{R}^n$ de rayon R et toute fonction $f \in L^p(B_R, w)$ prolongée par zéro hors de B_R. De plus, l'exposant $\frac{nq-\alpha}{nq(p-1)}$ de $A_p(w)$ est optimal, au sens où le résultat est faux si on remplace $\frac{nq-\alpha}{nq(p-1)}$ par un exposant plus petit.

Notre approche du Théorème 0.1 repose sur un argument de [4]. En effet, la démonstration du Théorème 0.1 s'appuie sur une combinaison d'une estimation de $I_\alpha f$ en fonction de Mf donnée dans [7] et de l'inégalité (2) ; on utilise une propriété des poids A_p établie dans [3]. Il est intéressant de remarquer que si on observe la dépendance exacte des quantités impliquées dans cet argument, on est conduit à la majoration optimal (5).

1. Introduction and result

Boundedness properties of various classical operators of harmonic analysis in weighted Lebesgue spaces can be characterized in terms of the membership of the relevant weights in the so called A_p classes. Recall that a weight w, namely a locally integrable nonnegative function in $\mathbb{R}^n$, is said to belong to the class A_p for some $p \in (1, \infty)$ if the quantity

$$A_p(w) = \sup_B \left(\frac{1}{|B|} \int_B w(x)\,dx \right) \left(\frac{1}{|B|} \int_B w(x)^{\frac{1}{1-p}}\,dx \right)^{p-1} \tag{6}$$

is finite. Here, B denotes any ball in $\mathbb{R}^n$ and $|B|$ stands for its Lebesgue measure. The number $A_p(w)$ is called the A_p-constant of w.

Introduced in [9], where the Hardy–Littlewood maximal operator is proved to be bounded in the Lebesgue space $L^p(\mathbb{R}^n, w)$, equipped with the measure $w(x)\,dx$, if and only if $w \in A_p$, the A_p condition has subsequently been shown to properly describe the class of weighted Lebesgue spaces between which the Hilbert transform and more general singular integral operators are bounded. For an account of more recent developments of the theory of A_p weights in harmonic analysis we refer to the monographs [14] and [6]. Let us just recall here that A_p weights play a key role in other branches of mathematical analysis, including the theory of degenerate elliptic equations, the related nonlinear potential theory, and the theory of quasiregular maps.

A question that remained open for quite a long time is that of an optimal estimate for the growth of the norm of the operators mentioned above in terms of the A_p-constant. The problem for the Hardy–Littlewood maximal operator M was solved in [2], where it is shown that, if $w \in A_p$ for some $p \in (1, \infty)$, then

$$\left(\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \right)^{\frac{1}{p}} \leqslant C A_p(w)^{\frac{1}{p-1}} \left(\int_{\mathbb{R}^n} \left| f(x) \right|^p w(x)\,dx \right)^{\frac{1}{p}} \tag{7}$$

for some constant $C = C(p, n)$ and for every locally integrable function f. Here, the power $\frac{1}{p-1}$ for $A_p(w)$ is the smallest possible, and hence the best one, since $A_p(w) \geqslant 1$. Very recently, analogous results have been established for the Hilbert transform in $L^p(\mathbb{R}, w)$ [11], and for the Riesz transforms in $L^p(\mathbb{R}^n, w)$ [12]. The exact growth of the norm of the Beurling–Ahlfors transform in weighted spaces has been determined in [13], and has consequences in the regularity of solutions of the Beltrami equation in the plane [1].

In the present Note we address a parallel issue for the fractional integral operator I_α, also called Riesz potential, of any order $\alpha \in (0, n)$. The Riesz potential of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}}\,dy \quad \text{for } x \in \mathbb{R}^n. \tag{8}$$

A very classical result states that I_α is bounded from $L^p(\mathbb{R}^n)$ into $L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n)$ if $1 < p < \frac{n}{\alpha}$. Weighted inequalities for fractional integrals can be characterized in terms of the A_p condition. As shown in [10], a necessary and sufficient condition for the boundedness of I_α from $L^p(\mathbb{R}^n, w)$ into $L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n, w)$ is that $w^{\frac{np}{n-\alpha p}} \in A_{1+\frac{q}{p'}}$, where $p' = \frac{p}{p-1}$. (See also [8] for an alternate treatment of weighted inequalities for potentials in terms of capacities.)

If w is merely in A_p for some $p \in (1, \frac{n}{\alpha})$, then there still exists a power $\overline{p}$, larger than p and depending on $A_p(w)$, such that I_α is bounded from $L^p(B, w)$ into $L^{\overline{p}}(B, w)$ for any ball $B \subset \mathbb{R}^n$ [5]. Our result provides sharp quantitative information on this statement, and reads as follows:

Theorem 1.1. *Let $n \geqslant 2$, $\alpha \in (0, n)$ and $1 < p < \frac{n}{\alpha}$. Let $w \in A_p$. Then, there exist positive constants $k = k(p, n)$ and $C = C(\alpha, p, n)$ such that if*

$$p - k A_p(w)^{\frac{1}{1-p}} < q \leqslant p \tag{9}$$

then

$$\left(\frac{1}{\int_{B_R} w(x)\,dx} \int_{B_R} \left| I_\alpha f(x) \right|^{\frac{nqp}{nq-\alpha p}} w(x)\,dx \right)^{\frac{nq-\alpha p}{nqp}} \leqslant C R^\alpha A_p(w)^{\frac{nq-\alpha}{nq(p-1)}} \left(\frac{1}{\int_{B_R} w(x)\,dx} \int_{B_R} \left| f(x) \right|^p w(x)\,dx \right)^{\frac{1}{p}} \tag{10}$$

for any ball $B_R \subset \mathbb{R}^n$ of radius R and every function $f \in L^p(B_R, w)$ (continued by 0 outside B_R). Moreover, the exponent $\frac{nq-\alpha}{nq(p-1)}$ for $A_p(w)$ is sharp, in the sense that the statement is false if $\frac{nq-\alpha}{nq(p-1)}$ is replaced by any smaller exponent.

The proof of Theorem 1 is related to an argument from [4]. In fact, the proof of Theorem 1 relies upon a combination of an estimate for $I_\alpha f$ in terms of Mf appearing in [7] with inequality (7), and exploits a property of A_p weights established in [3]. It is however interesting that keeping track of the exact dependence of the quantities involved in this argument can lead to the sharp bound (10).

2. Proof

Given any $\varepsilon > 0$, define

$$I_\alpha^\varepsilon f(x) = \int\limits_{\{|x-y|\geqslant\varepsilon\}\cap B_R} f(y)|x-y|^{\alpha-n}\,\mathrm{d}y \quad \text{for } x \in B_R. \tag{11}$$

A constant $C_1 = C_1(\alpha, p, n)$ exists such that

$$\left|I_\alpha f(x) - I_\alpha^\varepsilon f(x)\right| \leqslant C_1\varepsilon^\alpha Mf(x) \quad \text{for } x \in B_R, \tag{12}$$

see [7]. On the other hand, by Hölder's inequality,

$$\left|I_\alpha^\varepsilon f(x)\right| \leqslant \|f\|_{L^p(B_R,w)}\Bigg(\int\limits_{\{|x-y|\geqslant\varepsilon\}\cap B_R} |x-y|^{(\alpha-n)p'} w(y)^{\frac{1}{1-p}}\,\mathrm{d}y\Bigg)^{\frac{1}{p'}} \quad \text{for } x \in B_R. \tag{13}$$

Owing to a result from [3], a constant k as in the statement exists such that, if q fulfills (9), then $w \in A_q$ and

$$A_q(w) \leqslant C_2 A_p(w) \tag{14}$$

for some constant $C_2 = C_2(p, n)$. For any such q, another application of Hölder's inequality to the integral on the right-hand side of (13) yields

$$\left|I_\alpha^\varepsilon f(x)\right| \leqslant C_3\|f\|_{L^p(B_R,w)}\varepsilon^{\alpha-\frac{nq}{p}}\Bigg(\int\limits_{B_R} w(y)^{\frac{1}{1-q}}\,\mathrm{d}y\Bigg)^{\frac{q-1}{p}} \quad \text{for } x \in B_R, \tag{15}$$

for some constant $C_3 = C_3(\alpha, p, n)$. The choice $\varepsilon = \Big(\frac{Mf(x)}{\|f\|_{L^p(B_R,w)}(\int_{B_R} w(y)^{\frac{1}{1-q}}\,\mathrm{d}y)^{\frac{q-1}{p}}}\Big)^{-\frac{p}{nq}}$ in (12) and (15) yields

$$I_\alpha f(x) \leqslant C_4\big(Mf(x)\big)^{1-\frac{\alpha p}{nq}}\|f\|_{L^p(B_R,w)}^{\frac{\alpha p}{nq}}\Bigg(\int\limits_{B_R} w(y)^{\frac{1}{1-q}}\,\mathrm{d}y\Bigg)^{\frac{\alpha}{nq'}} \quad \text{for } x \in B_R, \tag{16}$$

for some constant $C_4 = C_4(\alpha, p, n)$. From (16) and (7) one infers that

$$\begin{aligned}\|I_\alpha f\|_{L^{\frac{nqp}{nq-\alpha p}}(B_R,w)} &\leqslant C_4\|Mf\|_{L^p(B_R,w)}^{\frac{nq-\alpha p}{nq}}\|f\|_{L^p(B_R,w)}^{\frac{\alpha p}{nq}}\Bigg(\int\limits_{B_R} w(y)^{\frac{1}{1-q}}\,\mathrm{d}y\Bigg)^{\frac{\alpha}{nq'}} \\ &\leqslant C_5 A_p(w)^{\frac{nq-\alpha p}{nq(p-1)}}\|f\|_{L^p(B_R,w)}\Bigg(\int\limits_{B_R} w(y)^{\frac{1}{1-q}}\,\mathrm{d}y\Bigg)^{\frac{\alpha}{nq'}},\end{aligned} \tag{17}$$

for some constant $C_5 = C_5(\alpha, p, n)$, whence, by the definition of $A_q(w)$, one obtains that

$$\left(\frac{1}{\int_{B_R} w(x)\,\mathrm{d}x}\int_{B_R}\left|I_\alpha f(x)\right|^{\frac{nqp}{nq-\alpha p}} w(x)\,\mathrm{d}x\right)^{\frac{nq-\alpha p}{nqp}} \leqslant C_6 R^\alpha A_p(w)^{\frac{nq-\alpha p}{nq(p-1)}} A_q(w)^{\frac{\alpha}{nq}}\left(\frac{1}{\int_{B_R} w(x)\,\mathrm{d}x}\int_{B_R}\left|f(x)\right|^p w(x)\,\mathrm{d}x\right)^{1/p}, \tag{18}$$

for some constant $C_6 = C_6(\alpha, p, n)$. Hence, inequality (10) follows, owing to (14).

In order to prove that the exponent $\frac{nq-\alpha p}{nq(p-1)}$ in (10) is sharp, consider a ball centered at 0 of any radius R, and functions f and weights w having the form $f(x) = \phi(\omega_n|x|^n)$ and $w(x) = \psi(\omega_n|x|^n)$ for $x \in \mathbb{R}^n$, for some functions $\phi, \psi : [0, \infty) \to [0, \infty)$, with ϕ vanishing outside $[0, \omega_n R^n)$. Here, ω_n denotes the Lebesgue measure of the unit ball. One has

$$I_\alpha f(x) \geqslant 2^{\alpha-n}\int_{\{|y|<|x|\}}\frac{f(y)}{|x|^{n-\alpha}}\,\mathrm{d}y = 2^{\alpha-n}|x|^{\alpha-n}\int_0^{\omega_n|x|^n}\phi(r)\,\mathrm{d}r \quad \text{for } x \in \mathbb{R}^n. \tag{19}$$

Thus, on setting $\omega_n R^n = t$, we get

$$\sup_f \frac{\left(\frac{1}{\int_{B_R} w(x)\,\mathrm{d}x}\int_{B_R}|I_\alpha f(x)|^{\frac{nqp}{nq-\alpha p}} w(x)\,\mathrm{d}x\right)^{\frac{nq-\alpha p}{nqp}}}{\left(\frac{1}{\int_{B_R} w(x)\,\mathrm{d}x}\int_{B_R}|f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}} \geqslant C_7 \sup_\phi \frac{\left(\int_0^t \psi(s)\,\mathrm{d}s\right)^{\frac{\alpha}{nq}}\left(\int_0^t\left(\int_0^s \phi(r)\,\mathrm{d}r\right)^{\frac{nqp}{nq-\alpha p}} s^{\frac{(\alpha-n)qp}{nq-\alpha p}}\psi(s)\,\mathrm{d}s\right)^{\frac{nq-\alpha p}{nqp}}}{\left(\int_0^t \phi(s)^p\psi(s)\,\mathrm{d}s\right)^{1/p}}$$
$$\geqslant C_7\left(\int_0^t \psi(s)\,\mathrm{d}s\right)^{\frac{\alpha}{nq}} \sup_{0<\tau<t}\left(\int_\tau^t r^{\frac{(\alpha-n)qp}{nq-\alpha p}}\psi(r)\,\mathrm{d}r\right)^{\frac{nq-\alpha p}{npq}}\left(\int_0^\tau \psi(r)^{\frac{1}{1-p}}\,\mathrm{d}r\right)^{\frac{1}{p'}} \tag{20}$$

for some constant $C_7 = C_7(\alpha, n)$, where the last inequality holds by a classical characterization of one-dimensional Hardy-type inequalities – see e.g. [8, Theorem 1.3.1/1].

Now, choose $\psi(s) = s^{(p-1)(1-\delta)}$ for $s > 0$, with $\delta \in (0, 1)$. It is then well known that $A_p(w) \approx \delta^{1-p}$ as $\delta \to 0^+$, up to multiplicative constants depending only on p and n [2]. Next, take $q = p - a\delta$, for sufficiently small a depending on k, in such a way that condition (9) is fulfilled. Computations show that, with these choices of ψ and q, the rightmost side of (20) $\approx \delta^{\frac{\alpha}{nq}-1}$ as $\delta \to 0^+$, up to multiplicative constants depending only on α, p and n. Since $A_p(w)^{\frac{nq-\alpha}{nq(p-1)}} \approx \delta^{\frac{\alpha}{nq}-1}$ as well, the sharpness of the exponent $\frac{nq-\alpha}{nq(p-1)}$ follows.

Acknowledgement

We wish to thank the referee for carefully reading the manuscript and for his/her valuable remarks on the original version.

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