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The explicit characterization of coefficients of a.e. convergent orthogonal series

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Abstract

We characterize sequences of numbers (a_n) such that $\sum_{n\geqslant 1} a_n\Phi_n$ converges a.e. for any orthonormal system (Φ_n) in any L_2-space. ***To cite this article: A. Paszkiewicz, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Caractérisation explicite des coefficients des séries orthogonales convergentes presques partout. On donne une complète caractérisation de la suite des nombres (a_n) telle que $\sum_{n\geqslant 1} a_n\Phi_n$ converge, presque partout, pour tout système orthogonal (Φ_n) dans tout espace $\mathbb{L}_2$.

La démonstration détaillées est donnée par A. Paszkiewicz dans l'article : On complete characterization of coefficients of a.e. convergent orthogonal series. ***Pour citer cet article : A. Paszkiewicz, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

This Note presents a complete characterization of sequences (a_n) for which:

(a) $\sum a_n\Phi_n$ converges a.e. for any orthonormal (O.N. for short) sequence (Φ_n) in any L_2-space.

The main result stated in Theorem 6 below is proved in [2]. Without loss of generality we consider only sequences (a_n) satisfying $a_n \geqslant 0$, $\sum_{n\geqslant 1} a_n^2 \leqslant 1$. Let such a sequence be fixed and let

$$A = \left\{\sum_{n\geqslant m} a_n^2;\ m = 1, 2, \ldots\right\}. \tag{1}$$

It is well known that a very sharp sufficient condition for (a) can be formulated by the use of so-called majorizing measures. We say, as in [2, Definition 1.7], that m is a *majorizing measure on A* if m is a Borel measure on $\mathbb{R}$ concentrated on A, and

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$$\int_0^1 \frac{d\epsilon}{\sqrt{m((t-\epsilon^2, t+\epsilon^2))}} \leqslant 1 \quad \text{for any } t \in A.$$

The existence of a finite majorizing measure on A implies the a.e.-convergence of $\sum a_n \Phi_n$ for any O.N.-system (Φ_n). This can be obtained from [3, Theorems 4.6 and 2.9], the details are explained in [6] and also in [2, Sections 2.9, 2.10].

Nevertheless, an explicit characterization of coefficients a_n, $n \geqslant 1$, satisfying (a) was an open problem for decades, as indicated by a number of authors (see V.F. Gaposhkin [1], M. Talagrand [4]).

A solution is presented in this Note. We show, in particular, that the existence of a finite majorizing measure on A is equivalent to condition (a), and we construct a majorizing measure $m_{\bar{A}}$ on the closure $\bar{A}$ with the smallest total mass $m_{\bar{A}}(\bar{A})$.

Our new formulas giving explicit characterizations of sequences (a_n) satisfying (a), are complicated. It is interesting to present them together with a simpler characterization of unconditional a.e.-convergence of series $\sum a_n \Phi_n$, announced in [2].

Definition 1. Let us denote $d_n^k = [\frac{n}{2^k}, \frac{n+1}{2^k})$, $0 \leqslant n < 2^k$, for $k \geqslant 1$, and let

$$\Delta_k^A = \bigcup_{n \in \Sigma_k} d_n^k$$

with

$$\Sigma_k = \{n = 0, \ldots, 2^k - 1;\ d_n^k \cap A \neq \emptyset\}, \quad k \geqslant 1.$$

By $\|\cdot\|$ we denote the L_2-norm in $L_2[0,1)$ or in another L_2-space of real functions, writing $\|h\| = \infty$ when $\int |h|^2 = \infty$. As usual, $1_Z(\cdot)$ is an indicator of the set Z.

Relatively simple characterizations can be formulated for a.e.-convergence of permutations of the series $\sum a_n \Phi_n$ in (a) as follows:

Theorem 2. *(See [2, Theorem 1.2].) The following conditions are equivalent*:

(b) *there exists a permutation σ on the set $\mathbb{N}$ of positive integers such that*

$$\sum_{n \geqslant 1} a_{\sigma(n)} \Phi_n \quad \textit{converges a.e. for any O.N.-system } (\Phi_n); \tag{2}$$

(β) $\|\sum_{k \geqslant 1} 1_{\Delta_k^A}\| < \infty$ *for A given by* (1).

Theorem 3. *(See [2, Theorem 1.3].) The following conditions are equivalent*:

(c) *for any permutation σ of $\mathbb{N}$,* (2) *is satisfied*;
(γ) $\sum_{k \geqslant 1} \|1_{\Delta_k^A}\| < \infty$ *for A given by* (1).

Obviously,

$$(c) \Longrightarrow (a) \Longrightarrow (b),$$

and thus any condition (α) equivalent to (a) should satisfy

$$(\gamma) \Longrightarrow (\alpha) \Longrightarrow (\beta).$$

It turns out that (α) can be obtained by the following more delicate analysis of the indicators $1_{\Delta_k^A}$:

Definition 4. For any $k \geqslant 1$, let $\mathcal{F}_k = \sigma(d_n^k; 0 \leqslant n < 2^k)$ be the σ-field generated by the intervals $d_n^k = [\frac{n}{2^k}, \frac{n+1}{2^k})$. By $\|h\|_k$ we denote the 'conditional L_2-norm'

$$\|h\|_k = \left(\mathbb{E}\left(h^2 | \mathcal{F}_k\right)\right)^{\frac{1}{2}}$$

for a real L_2-function h on [0, 1), where $\mathbb{E}(\cdot|\mathcal{F}_k)$ denotes the conditional expectation in [0, 1) with respect to Lebesgue measure λ. Thus $\|h\|_k$ is $\mathcal{F}_k$-measurable.

Definition 5. For L_2-functions $h : [0, 1) \to [0, \infty)$ we define (non-linear) operations

$$V_k^A h = 1_{\Delta_k^A} + \|h\|_k, \quad k \geqslant 1.$$

The main result can be formulated in the following way:

Theorem 6. *(See [2, Theorem 1.8].) For a sequence of coefficients (a_n), $\sum a_n^2 \leqslant 1$, the following conditions are equivalent*:

(a) $\sum_{n \geqslant 1} a_n \Phi_n$ *converges a.e. for any O.N. sequence* (Φ_n);
(A) *there exists a majorizing measure m on A with $m(A) < \infty$ for A given by* (1);
(α) $\lim_{l \to \infty} \|V_1^A \cdots V_l^A 0\| < \infty$.

If conditions (a), (A) *or* (α) *are not satisfied, then $\sum_{n \geqslant 1} a_n \Phi_n$ diverges a.e. for some O.N. sequence (Φ_n).*

If conditions (a), (A) or (α) are satisfied, then we can construct some canonical majorizing measure $m_{\bar{A}}$ on $\bar{A} = A \cup \{0\}$ with minimal total mass $m_{\bar{A}}(\bar{A})$. To do this we introduce the following operations:

Definition 7. For an L_2-function $h : [0, 1) \to [0, \infty)$ we define

$$W_k h = \frac{\|h\|_k + 1}{\|h\|_k} h$$

with the convention $\frac{a}{0} 0 = 0$ for $a \geqslant 0$. Let m_l^A be the measure on [0, 1) with density $\mathrm{d}m_l^A / \mathrm{d}\lambda = (W_1 \cdots W_{l-1} 1_{\Delta_l^A})^2$, $l \geqslant 1$, for Δ_l^A given by Definition 1.

Theorem 8. *(See [2, Theorem 8.11].) The measures m_l^A converge weakly, for $l \to \infty$, to some measure $m_{\bar{A}}$ concentrated on the closure $\bar{A}$ and $2m_{\bar{A}}$ is a majorizing measure on $\bar{A}$ with*

$$2m_{\bar{A}}(\bar{A}) \leqslant C \inf\{m(A);\ m - \text{a majorizing measure on } A\},$$

for some constant C.

Moreover, any majorizing measure $m_{\bar{A}}$ on $\bar{A}$ is a weak limit of a sequence of some majorizing measures on A [2, Proposition 1.9].

In fact for $M(A)$ and $N(A)$ being any two of the following three functions

$$A \mapsto \lim_{l \to \infty} \left\| V_1^A \cdots V_l^A 0 \right\|,$$

$$A \mapsto \lim_{l \to \infty} \|W_1 \cdots W_{l-1} 1_{\Delta_l^A}\| = \sqrt{m_{\bar{A}}(\bar{A})},$$

and

$$A \mapsto \sup_{\Phi_n - \text{O.N.-system}} \left\| \sup_{n \geqslant 1} |a_1 \Phi_1 + \cdots + a_n \Phi_n| \right\|,$$

defined for all sets A of the form (1), M and N are of the same ‘size’, i.e.,

$$\frac{1}{C} M(A) - C \leqslant N(A) \leqslant C M(A) + C$$

for some universal constant C.

Moreover, K. Tandori has proved (see [5] and [2, Theorems 8.4, 8.4*]) that for any O.N.-system (Φ_n) condition (a) is equivalent to

$$\Big\| \sup_{n\geqslant 1} |a_1\Phi_1 + \cdots + a_n\Phi_n| \Big\| < \infty.$$

The main difficulty in the proof of Theorem 6 is the construction, for a given finite sequence $(a_n)_{n\leqslant n\leqslant N}$, of a system $(\Phi_n)_{n\leqslant n\leqslant N}$ such that $\| \sup_{1\leqslant n\leqslant N} |a_1\Phi_1 + \cdots + a_n\Phi_n| \|$ is maximal possible. This is done in [2, Sections 3–7].

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