



Number Theory/Algebraic Geometry

Canonical subgroups over Hilbert modular varieties

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Abstract

We obtain new results on the geometry of Hilbert modular varieties in positive characteristic and morphisms between them. Using these results and methods of rigid geometry, we develop a theory of canonical subgroups for abelian varieties with real multiplication. *To cite this article: E.Z. Goren, P.L. Kassaei, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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Résumé

Sous-groupes canoniques sur les variétés modulaires de Hilbert. Nous obtenons des résultats nouveaux sur la géométrie des variétés modulaires de Hilbert en caractéristique positive et sur les morphismes entre celles-ci. Grâce à ces résultats et des méthodes de géométrie rigide, nous développons une théorie des sous-groupes canoniques pour les variétés abéliennes à multiplication réelle. *Pour citer cet article : E.Z. Goren, P.L. Kassaei, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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Soit p un nombre premier et « val » la valuation de \mathbb{C}_p satisfaisant $\text{val}(p) = 1$. Soit L/\mathbb{Q} un corps totalement réel de degré g , non ramifié en p , et \mathcal{O}_L son anneau d'entiers. Soit aussi κ un corps fini contenant les corps résiduels de tous les idéaux premiers de \mathcal{O}_L divisant p , $\mathbb{Q}_\kappa = W(\kappa)[1/p]$ et $\mathbb{B} = \text{Emb}(L, \mathbb{Q}_\kappa)$. On note σ l'automorphisme de Frobenius de \mathbb{Q}_κ , relevant le morphisme $x \mapsto x^p$ modulo p .

Soit $X/W(\kappa)$ la variété modulaire de Hilbert classifiant $\underline{A}/S = (A/S, \iota, \lambda, \alpha)$, où A est un schéma abélien sur un $W(\kappa)$ -schéma S , $\iota: \mathcal{O}_L \hookrightarrow \text{End}_S(A)$, λ une polarisation comme dans [5] et α une structure de niveau rigide et première à p . Soit $Y/W(\kappa)$ la variété modulaire de Hilbert classifiant $(\underline{A}/S, H)$, où \underline{A} est comme ci-dessus et H est un \mathcal{O}_L -sous-schéma en groupes plat, fini et isotrope de $A[p]$, de rang p^g . Notons $\pi: Y \rightarrow X$ le morphisme $(\underline{A}, H) \mapsto \underline{A}$. Les notations \bar{X} et $\mathfrak{X}_{\text{rig}}$ désignent respectivement la fibre spéciale de X et l'espace rigide associé au complété p -adique de X ; de même pour Y . Tous les morphismes induits sont notés π . Notons $\mathfrak{X}_{\text{rig}}^\circ$ le lieu de réduction ordinaire dans $\mathfrak{X}_{\text{rig}}$. Il y a alors une section de π , $s^\circ: \mathfrak{X}_{\text{rig}}^\circ \rightarrow \mathfrak{Y}_{\text{rig}}$, dont la réduction modulo p est donnée par $\underline{A} \mapsto (\underline{A}, \text{Ker}(\text{Frob}_A))$, fournissant ainsi un sous-groupe de A qui est un relèvement du noyau de Frobenius modulo p quand A est de réduction

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ordinaire. Le problème des sous-groupes canoniques peut être formulé comme suit : étendre s° à une section s^\dagger sur un ouvert admissible « maximal » contenant $\mathfrak{X}_{\text{rig}}^\circ$.

Pour $\beta \in \mathbb{B}$ et $P \in \mathfrak{X}_{\text{rig}}$, nous définissons $v_\beta(P) = \min\{1, \text{val}(\tilde{h}_\beta(P))\}$, où \tilde{h}_β est un relèvement local en P de l’invariant de Hasse partiel h_β de poids $p\sigma^{-1} \circ \beta - \beta$ défini dans [7]. Soit \mathcal{U} l’ouvert admissible de $\mathfrak{X}_{\text{rig}}$ contenant $\mathfrak{X}_{\text{rig}}^\circ$ et défini par : $v_\beta(P) + pv_{\sigma^{-1} \circ \beta}(P) < p$ pour tout $\beta \in \mathbb{B}$. Nous prouvons

Théorème 0.1. *Il existe une unique section $s^\dagger : \mathcal{U} \rightarrow \mathfrak{Y}_{\text{rig}}$ de π prolongeant $s^\circ : \mathfrak{X}_{\text{rig}}^\circ \rightarrow \mathfrak{Y}_{\text{rig}}$.*

Pour prouver le théorème, nous commençons par une étude détaillée de la géométrie globale de \bar{Y} en construisant une stratification sur \bar{Y} . Grâce à celle-ci, et en utilisant les paramètres fournis par la théorie de la déformation des variétés abéliennes, nous sommes en mesure de calculer le morphisme π infinitésimalement. Cela nous permet, en passant à la géométrie rigide, d’isoler une composante connexe \mathcal{V} de $\pi^{-1}(\mathcal{U})$. Nous prouvons que le degré du morphisme plat et fini $\pi|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{U}$ est égal à 1, et nous construisons s^\dagger en tant que $(\pi|_{\mathcal{V}})^{-1}$.

Pour $P = \underline{A} \in \mathcal{U}$, soit $s^\dagger(P) = (\underline{A}, H)$; nous appellerons H le sous-groupe canonique de \underline{A} . Nous prouvons aussi les résultats suivants, qui sont cruciaux dans les applications de la théorie :

Théorème 0.2. *Soit $\underline{A} \in \mathcal{U}$ défini sur K . Supposons qu’il existe $r \in K$ tel que $\text{val}(r) = \max_{\beta \in \mathbb{B}}\{v_\beta(\underline{A})\}$.*

- (i) *Le sous-groupe canonique de \underline{A} se réduit en $\text{Ker}(\text{Frob})$ modulo p/r .*
- (ii) *Pour tout C comme dans la définition de Y , il y a une méthode pour calculer $v_\beta(\underline{A}/C)$ en termes de $v_\beta(\underline{A})$.*

1. Introduction

The theory of the canonical subgroup originated with Lubin and Katz [12], initially motivated by defining the U operator for overconvergent elliptic modular forms. The power of such results became apparent in work on the Artin conjecture [3] and in results on analytic continuation of overconvergent modular forms as in [10,11], for example. Many authors have studied the canonical subgroup in various settings. We mention the works [1,2,4,6,13,15], as well as yet unpublished results by K. Buzzard, E. Nevens and J. Rabinoff. For applications as above one needs a refined theory of canonical subgroups. In this paper, we improve on the available literature in the case of Hilbert modular varieties, using a different technique than used by other authors; it continues our work in [8] and is based on the study of the special fiber of morphisms between the Shimura varieties in question and on deformation theory for abelian varieties. It thus seems suited to generalization to a wide class of Shimura varieties.

Notation. Let p be a prime number, L/\mathbb{Q} a totally real field of degree g in which p is unramified, \mathcal{O}_L its ring of integers, and $\mathfrak{N} \subseteq \mathcal{O}_L$ an ideal prime to p . For a prime $\mathfrak{p}|p$ let $\kappa_{\mathfrak{p}}$ denote the residue field, and let κ be a finite field containing every $\kappa_{\mathfrak{p}}$. Let \mathbb{Q}_κ be the fraction field of $W(\kappa)$. Let $\mathbb{B} = \text{Emb}(L, \mathbb{Q}_\kappa) = \coprod_{\mathfrak{p}} \mathbb{B}_{\mathfrak{p}}$, where \mathfrak{p} runs over prime ideals of \mathcal{O}_L dividing p , and $\mathbb{B}_{\mathfrak{p}} = \{\beta \in \mathbb{B} : \beta^{-1}(pW(\kappa)) = \mathfrak{p}\}$. Let σ denote the Frobenius automorphism of \mathbb{Q}_κ , lifting $x \mapsto x^p$ modulo p . It acts on \mathbb{B} via $\beta \mapsto \sigma \circ \beta$, and transitively on each $\mathbb{B}_{\mathfrak{p}}$. For $S \subseteq \mathbb{B}$ we let $\ell(S) = \{\sigma^{-1} \circ \beta : \beta \in S\}$, $r(S) = \{\sigma \circ \beta : \beta \in S\}$, and $S^c = \mathbb{B} - S$. The decomposition $\mathcal{O}_L \otimes_{\mathbb{Z}} W(\kappa) = \bigoplus_{\beta \in \mathbb{B}} W(\kappa)_\beta$, where $W(\kappa)_\beta$ is $W(\kappa)$ with the \mathcal{O}_L -action given by β , induces a decomposition $M = \bigoplus_{\beta \in \mathbb{B}} M_\beta$ on any $\mathcal{O}_L \otimes_{\mathbb{Z}} W(\kappa)$ -module M .

Let $X/W(\kappa)$ be the Hilbert modular variety classifying the data $\underline{A}/S = (A/S, \iota, \lambda, \alpha)$, where A is an abelian scheme over a $W(\kappa)$ -scheme S , $\iota : \mathcal{O}_L \hookrightarrow \text{End}_S(A)$, λ a polarization as in [5], and α a rigid $\Gamma_1(\mathfrak{N})$ -level structure. Let $Y/W(\kappa)$ be the Hilbert modular variety classifying $(\underline{A}/S, H)$, where \underline{A} is as above and H is a finite flat isotropic \mathcal{O}_L -subgroup scheme of $A[p]$ of rank p^g . Equivalently, Y is the moduli of $(f : \underline{A} \rightarrow \underline{B})$ where $\underline{A}, \underline{B}$, and $H = \text{Ker}(f)$ are as above. Let $\pi : Y \rightarrow X$ be the morphism $(\underline{A}, H) \mapsto \underline{A}$. Let $\bar{X}, \bar{\mathfrak{X}}, \bar{\mathfrak{X}}_{\text{rig}}$ be, respectively, the special fiber of X , the completion of X along \bar{X} , and the rigid analytic space associated to $\bar{\mathfrak{X}}$. We use a similar notation for Y and let π denote any of the induced morphisms. These spaces have models over \mathbb{Z}_p or \mathbb{Q}_p , denoted $X_{\mathbb{Z}_p}, \bar{\mathfrak{X}}_{\text{rig}, \mathbb{Q}_p}$, etc. For a point $P \in \bar{\mathfrak{X}}_{\text{rig}}$ we denote by $\bar{P} = \text{sp}(P)$ its specialization in \bar{X} , and similarly for Y . Let $w : Y \rightarrow Y$ be the automorphism $(\underline{A}, H) \mapsto (\underline{A}/H, A[p]/H)$. Let $s : \bar{X} \rightarrow \bar{Y}$ be the kernel-of-Frobenius section to π , $\underline{A} \mapsto (\underline{A}, \text{Ker}(\text{Fr} : A \rightarrow A^{(p)}))$.

We denote $s(\bar{X})$ by \bar{Y}_F , and $w(\bar{Y}_F)$ by \bar{Y}_V . These are components of \bar{Y} , and \bar{Y}_F (respectively, \bar{Y}_V) is the set of closed points (\underline{A}, H) where H is $\text{Ker}(\text{Fr}_A)$ (respectively, $\text{Ker}(\text{Ver}_A)$).

2. Geometric results

2.1. Global geometry of \bar{Y}

Let P be a closed point on \bar{X} corresponding to \underline{A} defined over a perfect field k of characteristic p . We define the type of \underline{A} to be $\tau(\underline{A}) = \{\beta \in \mathbb{B} : \mathbb{D}(\text{Ker}(\text{Fr}_A) \cap \text{Ker}(\text{Ver}_A))_\beta \neq 0\}$. Here \mathbb{D} is the contravariant Dieudonné module. For $(f : \underline{A} \rightarrow \underline{B})$ parameterized by Y corresponding to (\underline{A}, H) , we define $f^t : \underline{B} \rightarrow \underline{A}$ by the requirement $f \circ f^t = [p]$. We have $\bigoplus_{\beta \in \mathbb{B}} \text{Lie}(f)_\beta : \bigoplus_{\beta \in \mathbb{B}} \text{Lie}(\underline{A})_\beta \rightarrow \bigoplus_{\beta \in \mathbb{B}} \text{Lie}(\underline{B})_\beta$ and $\bigoplus_{\beta \in \mathbb{B}} \text{Lie}(f^t)_\beta : \bigoplus_{\beta \in \mathbb{B}} \text{Lie}(\underline{B})_\beta \rightarrow \bigoplus_{\beta \in \mathbb{B}} \text{Lie}(\underline{A})_\beta$. For $(f : \underline{A} \rightarrow \underline{B})$ parameterized by \bar{Y} , we define

$$\begin{aligned} \varphi(f) &= \varphi(\underline{A}, H) = \{\beta \in \mathbb{B} : \text{Lie}(f)_{\sigma^{-1} \circ \beta} = 0\}, & \eta(f) &= \eta(\underline{A}, H) = \{\beta \in \mathbb{B} : \text{Lie}(f^t)_\beta = 0\}, \\ I(f) &= I(\underline{A}, H) = \ell(\varphi) \cap \eta = \{\beta \in \mathbb{B} : \text{Lie}(f)_\beta = \text{Lie}(f^t)_\beta = 0\}. \end{aligned}$$

A pair (φ, η) of subsets of \mathbb{B} is called admissible if $\ell(\varphi)^c \subseteq \eta$. There are 3^s such pairs. We say $(\varphi', \eta') \geq (\varphi, \eta)$ if $\varphi' \supseteq \varphi$ and $\eta' \supseteq \eta$. For $\bar{Q} = (\underline{A}, H)$ on \bar{Y} , and $\bar{P} = \underline{A} = \pi(\bar{Q})$, the pair $(\varphi(\bar{Q}), \eta(\bar{Q}))$ is admissible and the following hold: $\varphi(\bar{Q}) \cap \eta(\bar{Q}) \subseteq \tau(\bar{P})$ and $(\varphi(\bar{Q}) - \eta(\bar{Q})) \cup (\eta(\bar{Q}) - \varphi(\bar{Q})) \subseteq \tau(\bar{P})^c$.

Definition 2.1. Let $\tau \subseteq \mathbb{B}$. We let W_τ denote the locally closed subset of \bar{X} whose closed points are \underline{A} of type τ . We let $Z_\tau = \bigcup_{\tau' \supseteq \tau} W_{\tau'}$. By [9], Z_τ is the Zariski closure of W_τ and $\{W_\tau\}$ form a stratification of \bar{X} by smooth quasi-affine varieties of dimension $\dim(W_\tau) = g - \sharp\tau$.

Let (φ, η) be admissible. We let $W_{\varphi, \eta}$ denote the locally closed subset of \bar{Y} whose closed points are \bar{Q} with $\varphi(\bar{Q}) = \varphi$ and $\eta(\bar{Q}) = \eta$. We let $Z_{\varphi, \eta} = \bigcup_{(\varphi', \eta') \geq (\varphi, \eta)} W_{\varphi', \eta'}$. For example $Z_{\mathbb{B}, \emptyset} = \bar{Y}_F$ and $Z_{\emptyset, \mathbb{B}} = \bar{Y}_V$.

For $\beta \in \mathbb{B}$ we let U_β^+ be the closed subset of \bar{Y} whose closed points are \bar{Q} with $\beta \in \varphi(\bar{Q})$. Similarly, V_β^+ is the closed subset whose points are \bar{Q} with $\beta \in \eta(\bar{Q})$.

Let \bar{P} be a closed point in \bar{X} defined over k . By [9] there is an isomorphism

$$\widehat{\mathcal{O}}_{X, \bar{P}} \cong W(k)[[t_\beta]]_{\beta \in \mathbb{B}}, \tag{1}$$

such that for $\tau \subseteq \tau(\bar{P})$ the image of Z_τ in $\text{Spf}(\widehat{\mathcal{O}}_{X, \bar{P}})$ is given by the ideal $\langle p, t_\beta : \beta \in \tau \rangle$. Let \bar{Q} be a closed point in \bar{Y} defined over k with $I(\bar{Q}) = I$. By [14] there is an isomorphism

$$\widehat{\mathcal{O}}_{Y, \bar{Q}} \cong W(k)[[x_\beta, y_\beta, z_\gamma]]_{\beta \in I, \gamma \in I^c} / \langle x_\beta y_\beta - p : \beta \in I \rangle. \tag{2}$$

The isomorphisms (1), (2) can be reduced modulo p to provide local parameters t_β at \bar{P} and $x_\beta, y_\beta, z_\beta$ at \bar{Q} . One can show that for Stamm's choice of isomorphism (2) we have the following: for every $\beta \in \varphi$ the image of U_β^+ (respectively, V_β^+) in $\text{Spf}(\widehat{\mathcal{O}}_{Y, \bar{Q}})$ is given by the ideal $\langle p, y_{\sigma^{-1} \circ \beta} \rangle$ (respectively, $\langle p, x_\beta \rangle$).

Theorem 2.2. Let (φ, η) be admissible, $I = \ell(\varphi) \cap \eta$. The following results hold:

- (i) $W_{\varphi, \eta}$ is non-empty and its Zariski closure is $Z_{\varphi, \eta}$. The collection $\{W_{\varphi, \eta}\}$ is a stratification of \bar{Y} .
- (ii) $W_{\varphi, \eta}$ is equidimensional and $\dim(W_{\varphi, \eta}) = 2g - (\sharp\varphi + \sharp\eta)$. We have $w(W_{\varphi, \eta}) = W_{r(\eta), \ell(\varphi)}$.
- (iii) The irreducible components of \bar{Y} are the irreducible component of the strata $Z_{\varphi, \ell(\varphi^c)}$ for $\varphi \subseteq \mathbb{B}$.
- (iv) Let \bar{Q} be a closed point of \bar{Y} with invariants (φ, η) . For an admissible (φ', η') such that $\varphi \supseteq \varphi' \supseteq \varphi - r(I)$ and $\eta \supseteq \eta' \supseteq \eta - I$ write $\varphi' = \varphi - J$, $\eta' = \eta - K$. Then, $\bar{Q} \in Z_{\varphi', \eta'}$ and $\widehat{\mathcal{O}}_{Z_{\varphi', \eta'}, \bar{Q}} = \widehat{\mathcal{O}}_{\bar{Y}, \bar{Q}} / \mathcal{I}$, where $\mathcal{I} = \langle x_\beta, y_\gamma : \beta \in I - K, \gamma \in I - \ell(J) \rangle$. In particular, every stratum in $\{Z_{\varphi, \eta}\}$ is nonsingular.

Proposition 2.1. Let (φ, η) be an admissible pair, and C an irreducible component of $Z_{\varphi, \eta}$.

- (i) The type is generically $\varphi \cap \eta$ on C ; furthermore, $C \cap \bar{Y}_F \cap \bar{Y}_V \neq \emptyset$.
- (ii) $\pi(W_{\varphi,\eta}) = \bigcup_{\tau'} W_{\tau'}$, where τ' is such that $\tau' \supseteq \phi \cap \eta$ and $\tau' - (\varphi \cap \eta) \subseteq \varphi^c \cap \eta^c$. Each fiber of $\pi|_{W_{\varphi,\eta}}$ is affine and irreducible of dimension $g - \sharp \varphi \cup \eta$; furthermore, $\pi(Z_{\varphi,\eta}) = Z_{\varphi \cap \eta}$.

2.2. Infinitesimal behavior of $\pi : \bar{Y} \rightarrow \bar{X}$

The following lemma plays a key role in our proof of the canonical subgroup theorem:

Lemma 2.3. *Let \bar{Q} be a closed point of \bar{Y} with invariants (φ, η) , $\bar{P} = \pi(\bar{Q})$, and $\pi^* : \widehat{\mathcal{O}}_{\bar{X},\bar{P}} \rightarrow \widehat{\mathcal{O}}_{\bar{Y},\bar{Q}}$ be the induced homomorphism. Let $\beta \in \varphi \cap \eta$. Using parameters as in (1) and (2), the following hold:*

- (i) If $\sigma \circ \beta \in \varphi$ and $\sigma^{-1} \circ \beta \in \eta$, then $\pi^*(t_\beta) = ux_\beta + vy_{\sigma^{-1} \circ \beta}^p$ for some units u, v in $\widehat{\mathcal{O}}_{\bar{Y},\bar{Q}}$.
- (ii) If $\sigma \circ \beta \in \varphi$ and $\sigma^{-1} \circ \beta \notin \eta$, then $\pi^*(t_\beta) = ux_\beta$ for some unit u in $\widehat{\mathcal{O}}_{\bar{Y},\bar{Q}}$.
- (iii) If $\sigma \circ \beta \notin \varphi$ and $\sigma^{-1} \circ \beta \in \eta$, then $\pi^*(t_\beta) = vy_{\sigma^{-1} \circ \beta}^p$ for some unit v in $\widehat{\mathcal{O}}_{\bar{Y},\bar{Q}}$.
- (iv) If $\sigma \circ \beta \notin \varphi$ and $\sigma^{-1} \circ \beta \notin \eta$, then $\pi^*(t_\beta) = 0$.

Furthermore, one can choose local parameters at $w(\bar{Q})$ as in (2) such that $w^* : \widehat{\mathcal{O}}_{\bar{Y},w(\bar{Q})} \rightarrow \widehat{\mathcal{O}}_{\bar{Y},\bar{Q}}$ is given by $w^*(x_\beta) = y_\beta$, $w^*(y_\beta) = x_\beta$, and $w^*(z_\beta) = z_\beta$.

3. Valuations and the canonical subgroup

3.1. The valuation cube

Let “val” be the valuation on \mathbb{C}_p satisfying $\text{val}(p) = 1$. We define $v(x) = \min\{\text{val}(x), 1\}$. Let $P \in \mathfrak{X}_{\text{rig}}$. Let $D_P = \text{sp}^{-1}(\bar{P})$ which is the rigid analytic space associated to $\text{Spf}(\widehat{\mathcal{O}}_{X,\bar{P}})$. The parameters t_β in (1) are functions on D_P . Similarly, for $Q \in \mathfrak{Y}_{\text{rig}}$, the parameters x_β in (2) are functions on $\text{sp}^{-1}(\bar{Q})$. We define $v_{\mathfrak{X}}(P) = (v_\beta(P))_{\beta \in \mathbb{B}}$ and $v_{\mathfrak{Y}}(Q) = (v_\beta(Q))_{\beta \in \mathbb{B}}$, where the entries $v_\beta(P), v_\beta(Q)$, are given by

$$v_\beta(P) = \begin{cases} v(t_\beta(P)), & \beta \in \tau(\bar{P}), \\ 0, & \beta \notin \tau(\bar{P}), \end{cases} \quad v_\beta(Q) = \begin{cases} 1, & \beta \in \eta(\bar{Q}) - I(\bar{Q}), \\ v(x_\beta(Q)), & \beta \in I(\bar{Q}), \\ 0, & \beta \notin \eta(\bar{Q}). \end{cases}$$

(Let h_β be the partial Hasse invariant which is a Hilbert modular form of weight $p\sigma^{-1} \circ \beta - \beta$, as in [7]. Then, in fact, $v_\beta(P) = v(\tilde{h}_\beta(P))$, where \tilde{h}_β is any lift of h_β locally at P .)

Proposition 3.1. *For any $Q \in \mathfrak{Y}_{\text{rig}}$, we have $v_{\mathfrak{Y}}(Q) + v_{\mathfrak{Y}}(w(Q)) = (1, 1, \dots, 1)$.*

Let $\Theta = [0, 1]^{\mathbb{B}}$ be the unit cube in $\mathbb{R}^{\mathbb{B}} \cong \mathbb{R}^g$. Its “open faces” can be encoded by vectors $\mathbf{a} = (a_\beta)_{\beta \in \mathbb{B}}$ such that $a_\beta \in \{0, *, 1\}$. The face $\mathcal{F}_{\mathbf{a}}$ corresponding to \mathbf{a} is the set $\{\mathbf{v} \in \Theta : v_\beta = a_\beta \text{ if } a_\beta \neq *, \text{ and } 0 < v_\beta < 1 \text{ otherwise}\}$. There are 3^g such faces. The star of an open face \mathcal{F} is $\text{Star}(\mathcal{F}) = \bigcup_{\mathcal{F}' \supseteq \mathcal{F}} \mathcal{F}'$, where the union is over all open faces \mathcal{F}' whose topological closure contains \mathcal{F} . For \mathbf{a} as above, we define $\eta(\mathbf{a}) = \{\beta \in \mathbb{B} : a_\beta \neq 0\}$, $I(\mathbf{a}) = \{\beta \in \mathbb{B} : a_\beta = *\}$, and $\varphi(\mathbf{a}) = r(\eta(\mathbf{a}))^c \cup I(\mathbf{a}) = r\{\beta \in \mathbb{B} : a_\beta \neq 1\}$.

Theorem 3.1. *There is a one-to-one correspondence between the open faces of Θ and the strata $\{W_{\varphi,\eta}\}$ of \bar{Y} , given by $\mathcal{F}_{\mathbf{a}} \mapsto W_{\phi(\mathbf{a}),\eta(\mathbf{a})}$. It has the following properties:*

- (i) $v_{\mathfrak{Y}}(Q) \in \mathcal{F}_{\mathbf{a}}$ if and only if $\bar{Q} \in W_{\phi(\mathbf{a}),\eta(\mathbf{a})}$.
- (ii) $\dim(W_{\phi(\mathbf{a}),\eta(\mathbf{a})}) = g - \dim(\mathcal{F}_{\mathbf{a}}) = \sharp\{\beta : a_\beta \neq *\}$.
- (iii) If $\mathcal{F}_{\mathbf{a}} \subseteq \mathcal{F}_{\mathbf{b}}$, then $W_{\phi(\mathbf{b}),\eta(\mathbf{b})} \subseteq \overline{W_{\phi(\mathbf{a}),\eta(\mathbf{a})}}$ and vice versa; that is, the correspondence is order reversing. In particular, $v_{\mathfrak{Y}}(Q) \in \text{Star}(\mathcal{F}_{\mathbf{a}}) \iff \bar{Q} \in Z_{\phi(\mathbf{a}),\eta(\mathbf{a})}$.

3.2. The canonical subgroup

Let $\bar{X}^\circ = W_\emptyset$ denote the ordinary locus in \bar{X} , and let $\bar{Y}^\circ = \pi^{-1}(\bar{X}^\circ)$ be the ordinary locus in \bar{Y} . Let $\mathfrak{X}_{\text{rig}}^\circ = \text{sp}^{-1}(\bar{X}^\circ)$, and $\mathfrak{Y}_{\text{rig}}^\circ = \text{sp}^{-1}(\bar{Y}^\circ)$; they can be shown to be affinoids. Let $\mathfrak{Y}_{\text{rig}}^{\circ\circ} = \text{sp}^{-1}(W_{\mathbb{B},\emptyset}) \subset \mathfrak{Y}_{\text{rig}}^\circ$. Then, $\mathfrak{Y}_{\text{rig}}^{\circ\circ}$ is an affinoid, and there is a canonical section $s^\circ : \mathfrak{X}_{\text{rig}}^\circ \rightarrow \mathfrak{Y}_{\text{rig}}^\circ$ to π whose image is $\overline{\mathfrak{Y}_{\text{rig}}^{\circ\circ}}$ and which satisfies $\overline{s^\circ(P)} = s^\circ(\bar{P})$.

Let

$$\mathcal{U} = \{P \in \mathfrak{X}_{\text{rig}}^\circ : v_\beta(P) + pv_{\sigma^{-1}\circ\beta}(P) < p, \forall \beta \in \mathbb{B}\}.$$

For a prime ideal \mathfrak{p} of \mathcal{O}_L dividing p , let $\mathcal{V}_\mathfrak{p} = \{Q \in \mathfrak{Y}_{\text{rig}}^\circ : v_\beta(Q) + pv_{\sigma^{-1}\circ\beta}(Q) < p, \forall \beta \in \mathbb{B}_\mathfrak{p}\}$, $\mathcal{W}_\mathfrak{p} = \{Q \in \mathfrak{Y}_{\text{rig}}^\circ : v_\beta(Q) + pv_{\sigma^{-1}\circ\beta}(Q) > p, \forall \beta \in \mathbb{B}_\mathfrak{p}\}$, and let $\mathcal{V} = \bigcap_{\mathfrak{p}|p} \mathcal{V}_\mathfrak{p}$, and $\mathcal{W} = \bigcup_{\emptyset \neq S \subseteq \{\mathfrak{p}|p\}} [\bigcap_{\mathfrak{p} \in S} \mathcal{W}_\mathfrak{p} \cap \bigcap_{\mathfrak{p} \notin S} \mathcal{V}_\mathfrak{p}]$. We can show that all these are admissible open subsets, and that \mathcal{U}, \mathcal{V} are connected.

The existence of canonical subgroups in the context of Hilbert modular varieties is a direct consequence of the following theorem:

Theorem 3.2. *We have $\pi(\mathcal{V}) = \mathcal{U}$. There is a section $s^\dagger : \mathcal{U} \rightarrow \mathfrak{Y}_{\text{rig}}^\circ$ to π whose image is \mathcal{V} . The section s^\dagger extends $s^\circ : \mathfrak{X}_{\text{rig}}^\circ \rightarrow \mathfrak{Y}_{\text{rig}}^{\circ\circ}$.*

Definition 3.3. Let $K \supseteq W(\kappa)$ be a completely valued field. Let \underline{A} be an abelian variety over K corresponding to $P \in \mathcal{U}$. Let $Q = s^\dagger(P)$ correspond to (\underline{A}, H) . We call H the canonical subgroup of \underline{A} .

The proof of Theorem 3.2 relies crucially on Lemma 2.3 which enables us to carry out a detailed analysis of the behavior of valuation vectors under the morphism π . This allows us to prove that $\pi^{-1}(\mathcal{U}) = \mathcal{V} \amalg \mathcal{W}$, as an admissible disjoint union in $\mathfrak{Y}_{\text{rig}}^\circ$. Consequently, $\pi|_{\mathcal{V}}$ is finite flat. Since \mathcal{V} is connected, the degree of $\pi|_{\mathcal{V}}$ is constant and can be calculated on $\pi^{-1}(\mathfrak{X}_{\text{rig}}^\circ) \cap \mathcal{V} = \mathfrak{Y}_{\text{rig}}^{\circ\circ}$, and, hence, is evidently equal to 1. It follows that $\pi|_{\mathcal{V}}$ is an isomorphism onto \mathcal{U} , and s^\dagger is constructed as $(\pi|_{\mathcal{V}})^{-1}$.

To describe the behavior of the canonical subgroup under certain isogenies, it is convenient to make the following definitions. Let $Q = (\underline{A}, H)$ be a point of $\mathfrak{Y}_{\text{rig}}^\circ$ and \mathfrak{p} a prime dividing p . We say that Q , or H , is *canonical at \mathfrak{p}* if $Q \in \mathcal{V}_\mathfrak{p}$; it is thus the canonical subgroup if it is canonical at every prime $\mathfrak{p}|p$. We say that Q , or H , is *anti-canonical at \mathfrak{p}* if $Q \in \mathcal{W}_\mathfrak{p}$ and we call it anti-canonical if $Q \in \bigcap_{\mathfrak{p}|p} \mathcal{W}_\mathfrak{p}$.

Theorem 3.4. *Let $K \supseteq W(\kappa)$ be a completely valued field. Let \underline{A}/K be an abelian variety corresponding to a point $P \in \mathcal{U}$. Let H be a subgroup of \underline{A} such that $(\underline{A}, H) \in \mathfrak{Y}_{\text{rig}}^\circ$. Enlarging K , we can assume the existence of $r \in K$ such that $\text{val}(r) = \max\{v_\beta(\underline{A}) : \beta \in \mathbb{B}\}$. Assume that H is canonical at \mathfrak{p} .*

- (i) *If $v_\beta(\underline{A}) + pv_{\sigma^{-1}\circ\beta}(\underline{A}) < 1$ for all $\beta \in \mathbb{B}_\mathfrak{p}$, then $v_\beta(\underline{A}/H) = pv_{\sigma^{-1}\circ\beta}(\underline{A})$ for all $\beta \in \mathbb{B}_\mathfrak{p}$ and $A[p]/H$ is anti-canonical at \mathfrak{p} . In particular, if H is canonical and $v_\beta(\underline{A}) + pv_{\sigma^{-1}\circ\beta}(\underline{A}) < 1$ for all $\beta \in \mathbb{B}$, then $A[p]/H$ is anti-canonical.*
- (ii) *If $1 < v_\beta(\underline{A}) + pv_{\sigma^{-1}\circ\beta}(\underline{A}) < p$ for all $\beta \in \mathbb{B}_\mathfrak{p}$, then $v_\beta(\underline{A}/H) = 1 - v_\beta(\underline{A})$ for all $\beta \in \mathbb{B}_\mathfrak{p}$ and $A[p]/H$ is canonical at \mathfrak{p} . In particular, if H is canonical and if $1 < v_\beta(\underline{A}) + pv_{\sigma^{-1}\circ\beta}(\underline{A}) < p$ for all $\beta \in \mathbb{B}$, then $A[p]/H$ is the canonical subgroup of \underline{A}/H .*
- (iii) *If there are $\mathfrak{p}|p$ and $\beta, \beta' \in \mathbb{B}_\mathfrak{p}$ such that $v_\beta(\underline{A}) + pv_{\sigma^{-1}\circ\beta}(\underline{A}) \leq 1$ and $v_{\beta'}(\underline{A}) + pv_{\sigma^{-1}\circ\beta'}(\underline{A}) \geq 1$, then $\underline{A}/H \notin \mathcal{U}$.*
- (iv) *Let C be a subgroup of \underline{A} which is anti-canonical at \mathfrak{p} . Then, $v_\beta(\underline{A}/C) = (1/p)v_{\sigma\circ\beta}(\underline{A})$, for all $\beta \in \mathbb{B}_\mathfrak{p}$, and $A[p]/C$ is canonical at \mathfrak{p} . In particular, if C is anti-canonical, then $A[p]/C$ is canonical.*
- (v) *The canonical subgroup of \underline{A} reduces to $\text{Ker}(\text{Fr})$ modulo p/r .*
- (vi) *Assume K contains $r^{1/p}$. Any anti-canonical subgroup of \underline{A} reduces to $\text{Ker}(\text{Ver})$ modulo $p/r^{1/p}$.*

The proof of assertions (1)–(4) is based on relating the valuation vectors $v_\mathfrak{X}(\underline{A})$, $v_\mathfrak{Y}(\underline{A}, H)$, $v_\mathfrak{Y}(w(\underline{A}, H)) = v_\mathfrak{Y}(\underline{A}/H, A[p]/H)$, and $v_\mathfrak{X}(\underline{A}/H)$, making use of Proposition 3.1, Lemma 2.3 and the behavior of valuation vectors under π . The proof of assertions (5), (6) uses Lemma 2.3.

4. Further results

- In order to carry out our arguments we found it convenient to use models X, Y over $W(\kappa)$ of $X_{\mathbb{Z}_p}, Y_{\mathbb{Z}_p}$. Nonetheless, by a descent argument that uses the functoriality of our construction relative to the Galois action, we can prove the following: *there exist open admissible subsets $\mathcal{U}_{\mathbb{Q}_p} \subset \mathfrak{X}_{\text{rig}, \mathbb{Q}_p}, \mathcal{V}_{\mathbb{Q}_p} \subset \mathfrak{Y}_{\text{rig}, \mathbb{Q}_p}$, and a section $s_{\mathbb{Q}_p}^\dagger : \mathcal{U}_{\mathbb{Q}_p} \rightarrow \mathcal{V}_{\mathbb{Q}_p}$ which becomes $s^\dagger : \mathcal{U} \rightarrow \mathcal{V}$ after extension of scalars from \mathbb{Q}_p to \mathbb{Q}_κ .*
- Let $L \subseteq M$ be a finite extension of totally real fields. The canonical subgroup is functorial relative to the natural morphism of moduli spaces $X^L \rightarrow X^M, Y^L \rightarrow Y^M$, which, in turn, allows us to prove, using [8], that our construction of the canonical subgroup is optimal in the following sense: *Let $\mathcal{T} \subseteq \mathfrak{X}_{\text{rig}}$ be an affinoid containing $v_x^{-1}((p/(p+1), \dots, p/(p+1)))$. There is no section to π defined on \mathcal{T} .*
- We can extend our theory of canonical subgroups to the minimal compactification. The results described above have a natural extension to these situations.
- We believe that our results on the geometry of Hilbert modular varieties and our refined theory of canonical subgroups will be instrumental in applications to analytic continuation of p -adic modular forms; this is work in progress. For instance, one can show that an overconvergent Hilbert modular eigenform of finite slope, which is defined over a strict neighborhood of $\mathfrak{Y}_{\text{rig}}^{\circ\circ}$, can be extended to \mathcal{V} .

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