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A posteriori error analysis of the heterogeneous multiscale method for homogenization problems

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Abstract

In this Note we derive a posteriori error estimates for a multiscale method, the so-called heterogeneous multiscale method, applied to elliptic homogenization problems. The multiscale method is based on a macro-to-micro formulation. The macroscopic method discretizes the physical problem in a macroscopic finite element space, while the microscopic method recovers the unknown macroscopic data on the fly during the macroscopic stiffness matrix assembly process. We propose a framework for the analysis allowing to take advantage of standard techniques for a posteriori error estimates at the macroscopic level and to derive residual-based indicators in the macroscopic domain for adaptive mesh refinement. ***To cite this article: A. Abdulle, A. Nonnenmacher, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Analyse a posteriori de la discrétisation d'un schéma multi-échelles pour des problèmes d'homogénéisation. Dans cette Note, nous proposons une analyse a posteriori d'un schéma multi-échelles de type « micro–macro » pour des problèmes d'homogénéisation. Les paramètres du schéma macroscopique, inconnus à priori, sont obtenus pendant l'assemblage du problème homogénéisé à l'aide de schémas microscopiques. Le cadre que nous proposons pour l'analyse du schéma multi-échelles nous permet d'utiliser des techniques standards pour obtenir des indicateurs a posteriori par résidu de l'erreur. Ces indicateurs d'erreur permettent de mettre en oeuvre une stratégie d'adaptation du maillage. ***Pour citer cet article : A. Abdulle, A. Nonnenmacher, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

Il est bien connu que l'adaptation du maillage permet une grande amélioration de la discrétisation par éléments finis d'équations aux dérivées partielles [16,18]. Cette adaptation du maillage repose sur l'analyse a posteriori de la discrétisation. Alors qu'il existe de nombreux travaux traitant de l'analyse a posteriori pour des discrétisations par éléments finis « classiques », peu de travaux traitent de cette analyse pour des schémas multi-échelles. Dans cette Note, nous proposons une analyse a posteriori d'un schéma multi-échelles appliqué à des problèmes d'homogénéisation.

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Ce schéma combine une méthode macroscopique, capable d'approximer la solution du problème homogénéisé, et des schémas microscopiques mis en oeuvre sur des micro-cellules contenues dans le maillage macroscopique. Nous montrons que les solutions des schémas microscopiques (utilisées pour le calcul du problème macroscopique) peuvent aussi servir à définir des sauts de flux macroscopiques, ingrédients cruciaux pour l'analyse a posteriori de l'erreur. La construction de ces flux, déjà utilisée dans [3] pour définir des schémas multi-échelles de type Galerkin discontinu, est à la base d'une formule de représentation de l'erreur permettant d'utiliser des techniques standards pour obtenir des indicateurs a posteriori par résidu de l'erreur. Nous montrons que notre estimation a posteriori de l'erreur est optimale, c'est-à-dire que nous obtenons une majoration et une minoration explicites de l'erreur par une quantité ne dépendant que des paramètres de la discrétisation, de la solution numérique et des données initiales du problème.

1. Introduction

This Note is concerned with a posteriori error estimates for a finite element multiscale method, the so-called finite element heterogeneous multiscale method (FE-HMM). This method has been introduced in [11]. Semi-discrete a priori error analysis for elliptic problems was performed in [13] and fully discrete analysis for various types of FE was obtained in [1,2,5,3]. The FE-HMM has been applied successfully to a variety of applications (see [4] for an overview). While numerous multiscale (or upscaling) FE methods have been constructed in recent years (see the references in [4]), very few rigorous a posteriori error analyses have been derived for such methods and adaptive strategies for multiscale methods are in turn still under-developed. We note that for elliptic problems, a first a posteriori error analysis (for multiscale problems) was given in [17]. In the aforementioned paper, the numerical method is obtained by a reformulation of the HMM for elliptic problems in a two-scale framework. However, a posteriori error estimates in the physical domain have not been derived and the analysis relies on a two-scale norm. Furthermore, the algorithm relies on an explicit decomposition of the oscillating tensor in fast and slow scale and requires in turn more regularity on the oscillating coefficients than coercivity and boundedness. Our algorithm relies on the macro–micro version of the HMM as proposed in [13,1,2,5,3] and can be applied to general oscillating diffusion tensors. Scale separation must be assumed in order for our strategy (relying on representative elements or sampling domains) to make sense. Rigorous a posteriori error bounds will be derived assuming only coercivity and boundedness of the *original oscillating tensor*. Up to a new data approximation term, our estimators are similar to the usual ones (for single scale problems). With more regularity, this data approximation term can also be estimated. We close this introduction by mentioning that for macro-to-micro methods another type of adaptivity is also important, namely the adaptivity of the sampling domain sizes. Finding adaptively the optimal spatial localization has been investigated by a few authors, we mention [15] for the variational multiscale framework and [12] for the HMM. Numerical experiments for the FE-HMM related to that question (for deterministic and random problems) can also be found in [6,4].

We consider second-order elliptic partial differential equations with highly oscillating coefficients on a domain $\Omega \in \mathbb{R}^d$, $d = 1, 2, 3$. For simplicity, we choose zero Dirichlet boundary conditions, but we emphasize that our a posteriori estimates can be derived for more general problems (e.g. non-zero Dirichlet or Neumann boundary conditions). We thus consider

$$-\nabla \cdot \left(a^\varepsilon \nabla u^\varepsilon\right) = f \quad \text{in } \Omega, \qquad u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where a^ε is symmetric, satisfies $a^\varepsilon(x) \in (L^\infty(\Omega))^{d\times d}$ and is uniformly elliptic and bounded, i.e. there exist $\lambda, \Lambda > 0$ such that $\lambda|\xi|^2 \leqslant a^\varepsilon(x)\xi \cdot \xi \leqslant \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and all ε, where ε represents a small scale in the problem that characterizes the multiscale nature of the tensor $a^\varepsilon(x)$.

An application of Lax–Milgram theorem gives us a family of solutions $\{u^\varepsilon\}_\varepsilon$ which is bounded in $H_0^1(\Omega)$ independently of ε. Furthermore, using the notions of G-convergence introduced by De Girorgi and Spagnolo (see for example [14, Chap. 5]), one can show that there exists a symmetric tensor $a^0(x)$ and a subsequence of $\{u^\varepsilon\}_\varepsilon$ which weakly converges to an element $u^0 \in H_0^1(\Omega)$, where u^0 is the solution of the so-called homogenized or upscaled problem

$$-\nabla \cdot \left(a^0 \nabla u^0\right) = f \quad \text{in } \Omega, \qquad u^0 = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $a^0(x)$ (the homogenized tensor) again satisfies $\lambda|\xi|^2 \leqslant a^0(x)\xi \cdot \xi \leqslant \Lambda|\xi|^2$, $\forall \xi \in \mathbb{R}^d$.

Let $\mathcal{T}_H$ be a shape regular partition of Ω in simplicial elements K of diameter H_K (the macro triangulation). We further choose a quadrature formula (QF) $\{\omega_{\hat{K}_\ell}, \hat{x}_\ell, \ell = 1, \ldots, \mathcal{L}\}$ on the reference triangle $\hat{K}$ which induces a QF over an arbitrary element $K = F_K(\hat{K})$ through the C^1-diffeomorphism F_K ($x_{K_{\delta_\ell}} = F_K(\hat{x}_\ell)$, $\omega_{K_\ell} = \omega_{\hat{K}_\ell} \det \partial F_K$). For each macro triangle K and each quadrature point $x_\ell \in K$ we define a sampling domain $K_{\delta_\ell} = x_{K_{\delta_\ell}} + \delta(-1/2, 1/2)^d$. The FE-HMM that we consider in this Note aims at capturing the above homogenized solution u^0 without relying on precomputing the homogenized tensor a^0. It relies on a *macro bilinear form*

$$B(v^H, w^H) = \sum_{K \in \mathcal{T}_H} \sum_{\ell=1}^{\mathcal{L}} \frac{\omega_{K_\ell}}{|K_{\delta_\ell}|} \int_{K_{\delta_\ell}} a^\varepsilon(x) \nabla v^h_{K_\ell} \cdot \nabla w^h_{K_\ell} \, \mathrm{d}x, \tag{3}$$

where $v^H, w^H \in V^p_H(\Omega, \mathcal{T}_H) = \{v^H \in H^1_0(\Omega);\ v^H|_K \in \mathcal{R}^p(K),\ \forall K \in \mathcal{T}_H\}$ is a finite dimensional subspace of $H^1_0(\Omega)$, and $\mathcal{R}^p$ is the space $P^p(K)$ of polynomials on K of total degree at most p if K is a simplicial FE, or the space $Q^p(K)$ of polynomials on K of degree at most p in each variables if K is a rectangular FE. Here and in what follows, $|K|$ and $|K_{\delta_\ell}|$ denote the measure of the domains K and K_{δ_ℓ}, respectively. The size of K (the macro elements) is usually much larger than the smallest scale ε of the problem, while the size of K_{δ_ℓ} (the sampling domains) is usually comparable to the ε length scale. This scale is accounted for in the micro-functions $v^h_{K_\ell}, w^h_{K_\ell}$ needed to assemble the bilinear form (3) and defined on K_{δ_ℓ}. These functions are obtained as follows: find $v^h_{K_\ell}$ such that $(v^h_{K_\ell} - v^H_{lin,K_\ell}) \in S^q(K_{\delta_\ell}, \mathcal{T}_h)$ and

$$\int_{K_{\delta_\ell}} a^\varepsilon(x) \nabla v^h_{K_\ell} \cdot \nabla z^h \, \mathrm{d}x = 0, \quad \forall z^h \in S^q(K_{\delta_\ell}, \mathcal{T}_h), \tag{4}$$

where $v^H_{lin,K_\ell}(x) = v(x_{K_{\delta_\ell}})^H + (x - x_{K_{\delta_\ell}}) \cdot \nabla v^H(x_{K_{\delta_\ell}})$ is a linearization of the macro-function v^H at the integration point $x_{K_{\delta_\ell}}$ (see [13,1,4] for details). Here $S^q_h(K_{\delta_\ell}, \mathcal{T}_h) = \{z^h \in W(K_{\delta_\ell});\ z^h|_T \in \mathcal{R}^q(T),\ T \in \mathcal{T}_h\}$ is the micro FE space, and the space $W(K_{\delta_\ell})$ determines the coupling condition. Several choices are possible for this latter space, for example $W(K_{\delta_\ell}) = H^1_0(K_{\delta_\ell})$ or $W(K_{\delta_\ell}) = W^1_{per}(K_{\delta_\ell})$, where $W^1_{per}(K_{\delta_\ell}) = H^1_{per}(K_{\delta_\ell})/\mathbb{R}$ and $H^1_{per}(K_{\delta_\ell})$ is the closure of infinitely differentiable periodic functions in K_{δ_ℓ} for the H^1 norm.

The macro-solution of the FE-HMM is defined by the following variational problem: find $u^H \in V^p_H(\Omega, \mathcal{T}_H)$ such that

$$B(u^H, v^H) = \int_\Omega f v^H \, \mathrm{d}x, \quad \forall v^H \in V^p_H(\Omega, \mathcal{T}_H). \tag{5}$$

Assuming $u^0 \in H^{p+1}$ and that the solution of (4) (in the exact Sobolev space $W(K_{\delta_\ell})$) is in $C(\Omega, H^{q+1}(K_{\delta_\ell}))$, one can for example show that $\|u^0 - u^H\|_{H^1(\Omega)} \leqslant C(H^p + (\frac{h}{\varepsilon})^{2q} + m_e)$ or $\|u^0 - u^H\|_{L^2(\Omega)} \leqslant C(H^{p+1} + (\frac{h}{\varepsilon})^{2q} + m_e)$. Here, H represents the macroscopic mesh size, h the microscopic mesh size and m_e a so-called modeling error which can be estimated in some cases, but which does not depend on the mesh sizes H or h. For example, if $a^\varepsilon(x) = a(x, x/\varepsilon)$, and K_{δ_ℓ} is chosen such that $\delta_\ell = \varepsilon$ and one collocates the tensor $a(x, x/\varepsilon)$ in (3) and (4) at macro quadrature points $a(x_\ell, x/\varepsilon)$, $x_\ell \in K$ then $m_e \equiv 0$ (see [13,1,2,5,3] and [4] for a comprehensive review).

Remark 1. Numerical experiments [1] show that the above bounds are sharp. As a consequence, a simultaneous refinement of macro and micro meshes is needed. For the case of piecewise linear FE approximation in macro- and micro-problems on a shape regular mesh, this refinement has to be done as $\hat{h} \propto H$ (L^2 norm) and $\hat{h} \propto \sqrt{H}$ (H^1 norm), respectively, where $H = \mathcal{O}((N_{mac})^{-(1/d)})$ and the scaled micro mesh size is $\hat{h} = h/\varepsilon = \varepsilon \cdot (N_{mic})^{-(1/d)}/\varepsilon = (N_{mic})^{-(1/d)}$ where $h = \varepsilon(N_{mic})^{-(1/d)}$ discretizes the small scale (here N_{mac} and N_{mic} denote the macro and the micro degrees of freedom, respectively).

2. Residual based a posteriori error estimates

In this section we present a posteriori error estimates for the FE-HMM. Our goal is to compute localized residuals which are used to adapt the macro mesh according to potential *macro* singularities of the coarse solution. These singularities may be caused e.g. by reentrant corners or high contrast in macroscopic coefficients. As the usual macroscopic data to compute the residuals are not at hand or precomputed in our method, we have to recover them from suitably averaged microscopic quantities. We show that this can be done with minimal overhead during the assembly of the FE method for a given mesh. For simplicity, we discuss here the case of piecewise linear macro and micro FE, with simplicial macro elements. In this case $\mathcal{L} = 1$, $x_{K_{\delta_\ell}} = x_{K_\delta}$ is located at the barycenter of K and $\omega_{K_\ell} = \omega_K = |K|$. We also set $\delta = \varepsilon$ and choose $S_h^1(K_\varepsilon, \mathcal{T}_h) \subset W_{per}^1(K_\varepsilon)$. Generalization of our a posteriori analysis to higher order FE, quadrilateral macro FE or to the case $\delta \neq \varepsilon$ could be obtained combining the framework sketched below, the results in [13] and the available theory for higher order FE-HMM (see [4] and the references therein). We start with a conformal macro mesh $\mathcal{T}_H$ and denote by $\mathcal{E}_H$ the set of interior interfaces. For an interface $e \in \mathcal{E}_H$ we denote by K^+ and K^- the two elements such that $e = K^+ \cap K^-$. For the solution $u^H \in V_H^1(\Omega, \mathcal{T}_h)$ of problem (5), we consider its corresponding micro-functions $u^{h,+}$ and $u^{h,-}$, which are solutions of (4) in the sampling domains K_ε^+ and K_ε^- of K^+ and K^-, respectively, constrained by the solution u^H of (5). To construct our residual-based a posteriori estimators, we will need the following "multiscale flux"

$$[\![\overline{a^\varepsilon(x)\nabla u^h}]\!]_e := \begin{cases} \left(\frac{1}{|K_\varepsilon^+|}\int_{K_\varepsilon^+} a^\varepsilon(x)\nabla u^{h,+}\,\mathrm{d}x - \frac{1}{|K_\varepsilon^-|}\int_{K_\varepsilon^-} a^\varepsilon(x)\nabla u^{h,-}\,\mathrm{d}x\right)\cdot n_e & \text{for } e \notin \partial\Omega, \\ 0 & \text{for } e \in \partial\Omega, \end{cases} \tag{6}$$

where n_e denotes the unit outward normal chosen to be $n_e = n_e^+$. This flux, first introduced in [3] in the context of multiscale discontinuous Galerkin methods, is the crucial building block to derive our estimates. We emphasize that these fluxes can be obtained with minimal overhead from the solution of the micro-problems (5) (needed for the assembly of the macrobilinear form) and the macro-solution u^H. Indeed, let $\{\varphi_{i,K}^H\}$ be the basis functions of $V_H^1(\Omega, \mathcal{T}_H)$ and $\varphi_{i,K}^h$ the corresponding micro-solutions. Then for every micro-problem we can compute and store $[\![\overline{a^\varepsilon(x)\nabla\varphi_{i,K}^h}]\!]_e$ with minimal extra cost; multiplication with u^H gives the appropriate multiscale flux. For each vector $\mathbf{e}_i$, $i = 1, \ldots, d$ of the canonical basis $\mathbb{R}^d$ we consider $\psi_{K_\varepsilon}^{i,h} \in S_h^1(K_\delta, \mathcal{T}_h)$ the solution of the micro problem (4) with right hand side $-\int_{K_\varepsilon} a^\varepsilon(x)\mathbf{e}_i \cdot \nabla z^h\,\mathrm{d}x$. We define the numerically homogenized tensor (constant on each macro element K) by

$$a_K^0 = \frac{1}{|K_\varepsilon|}\int_{K_\varepsilon} a^\varepsilon(x)\left(I + J_{\psi_{K_\varepsilon}^h(x)}^T\right)\mathrm{d}x, \tag{7}$$

where $J_{\psi_{K_\varepsilon}^h(x)}$ is a $d \times d$ matrix with entries $(J_{\psi_{K_\varepsilon}^h(x)})_{ij} = (\partial\psi_{K_\varepsilon}^{i,h})/(\partial x_j)$. Notice that the tensor (7) is never computed explicitly in the FE-HMM and it will only be used as a tool to derive the a posteriori error bounds. We define the *local error indicator* $\eta_H(K)$ on an element K by $\eta_H(K)^2 := H_K^2\|f_H\|_{L^2(K)}^2 + \frac{1}{2}\sum_{e\subset\partial K} H_e\|[\![\overline{a^\varepsilon(x)\nabla u^h}]\!]_e\|_{L^2(e)}^2$, where f_H is a piecewise constant approximation of f. Furthermore, we define the *data approximation error* $\xi_H(K)$ on an element K by $\xi_H(K)^2 := H_K^2\|f_H - f\|_{L^2(K)}^2 + \|(a_K^0 - a^0(x))\nabla u^H\|_{L^2(K)}^2$, where a^0 is the (unknown) tensor of the upscaled problem (2). Both the error indicator and the data approximation error are defined on a subset $\omega \subset \Omega$ by summing over all elements $K \subset \omega$. Following [8, App. A] one can show

Lemma 2.1.

$$\frac{1}{|K_\varepsilon|}\int_{K_\varepsilon} a^\varepsilon(x)\nabla v_K^h \cdot \nabla w_K^h\,\mathrm{d}x = \frac{1}{|K|}\int_K a_K^0\nabla v^H \cdot \nabla w^H\,\mathrm{d}x,$$

where v_K^h, w_K^h are the solutions of (4) *in $S_h^1(K_\varepsilon, \mathcal{T}_h)$ constrained by $v^H, w^H \in V_H^1(\Omega, \mathcal{T}_H)$.*

We define the error between the solution u^0 of (2) and the solution u^H of the problem (5) as $e^H := u^0 - u^H$. Let us further denote by $B_0(\cdot,\cdot)$ the bilinear form corresponding to (2). The following error representation formula is crucial for the derivation of our estimates and can be obtained using Lemma 2.1.

Lemma 2.2. *For all $v \in H_0^1(\Omega)$, we have*

$$B_0(e^H, v) = \int_\Omega fv\,\mathrm{d}x - \sum_{e\in\mathcal{E}}\int_e [\![\overline{a^\varepsilon(x)\nabla u^h}]\!]_e v\,\mathrm{d}s + \sum_{K\in\mathcal{T}_H}\int_K \left(a_K^0 - a^0(x)\right)\nabla u^H\nabla v\,\mathrm{d}x, \tag{8}$$

where u^H is the solution of (5)*, u^h are the corresponding micro-solutions of* (4) *defined on each sampling domain $K_\varepsilon \subset K$, and $[\![\overline{a^\varepsilon(x)\nabla u^h}]\!]_e$ is defined according to* (6)*.*

By setting $v = e^H$ in formula (8) and using the Clément interpolation, the following a posteriori upper bound can be obtained (see [7] for details).

Theorem 2.3. *(A posteriori upper bound.) There exists a constant $C > 0$ depending only on the shape regularity constant γ and the coercivity and continuity bound λ, Λ of a^ε such that $\|u - u_H\|^2_{H^1(\Omega)} \leqslant C(\eta_H(\Omega)^2 + \xi_H(\Omega)^2)$.*

Using test functions involving bubble functions (in each element K and edge e) in formula (8) we obtain the following a posteriori lower bound (see [7] for details).

Theorem 2.4. *(A posteriori lower bound.) There exists a constant $C > 0$ depending only on the shape regularity constant γ and the coercivity and continuity bound λ, Λ of a^ε such that $\eta_H(K)^2 \leqslant C(\|u - u_H\|^2_{H^1(\omega_K)} + \xi_H(\omega_K)^2)$, where the domain ω_K consists of all elements sharing at least one side with K.*

We emphasize that the above two theorems have been derived without *any structure assumptions* on the form of the coefficients (as e.g. periodicity) and with minimal regularity. As we here assume macroscopic singularities, we will not discuss a posteriori estimators for the micro-problem (4). Notice that the geometry of the sampling domain is simple (square) and the boundary conditions are periodic, thus singularities can only arise due to the singularities in the microscopic coefficients. In this latter case (as macro and micro calculations are separated in the assembly of the bilinear form (3)), standard a posteriori error estimates could be used for the micro-solutions. We notice, however, that our estimator η_H also depends on the micro-solution (4). The following lemma gives an estimation of the data approximation error in ξ_H and indicates how the micro mesh should be adapted to the macro mesh refinement (see [7] for details).

Lemma 2.5. *Assume $a^\varepsilon(x) = a(x, x/\varepsilon) = a(x, y)$ is periodic in the y variable and that $a(x, y) \in W^{1,\infty}(\bar{\Omega}; W^{1,\infty}_{per}(Y))$. Let $a^0(x)$ be the tensor of the homogenized problem* (2). *Then*

$$\left|a^0_{ij}(x) - a^0_{Kij}\right|_{L^\infty(\Omega)} \leqslant C\left(H + \varepsilon + \left(\frac{h}{\varepsilon}\right)^2\right), \tag{9}$$

where C depends only on Ω and the coercivity and continuity bound λ, Λ of a^ε.

The three terms of the above error can be decomposed as $|a^0_{ij}(x) - a^0_{K_{ij}}|_{L^\infty(\Omega)} \leqslant C(|a^0_{ij}(x) - a^0_{ij}(x_K)|_{L^\infty(\Omega)} + |a^0_{ij}(x_K) - \tilde{a}^0_{K_{ij}}|_{L^\infty(\Omega)} + |a^0_{K_{ij}} - \tilde{a}^0_{K_{ij}}|_{L^\infty(\Omega)})$, where x_K is the quadrature point in each macro element K, and $\tilde{a}_K$ is defined similarly as (7) but with exact micro-functions ψ^j (solution of (4) in $W^1_{per}(K_\varepsilon)$). The first term of the right-hand side of (9) can be estimated as CH (macro error), the second term as $C\varepsilon$ (modeling error) and the last term as $C(\frac{h}{\varepsilon})^2$ (micro error). By collocating the slow variable in the bilinear form (3), in the micro-problems (4) and in the multiscale flux (6) at the quadrature point (in case of an explicitly decomposable oscillating tensor $a^\varepsilon = a(x, x/\varepsilon)$) the modeling error vanishes. If we further choose a tensor of the form $a^\varepsilon = a(x/\varepsilon)$ the macro error also vanishes. Finally, for a tensor of the form $a^\varepsilon = a(x/\varepsilon)$ and if we assume exact micro-functions, we recover the classical residual based a posteriori error bounds.

Remark 2. We deduce from (9) that the micro mesh should be adapted to the macro mesh refinement as $\hat{h} \simeq \sqrt{H}$ in order for the data approximation to be of comparable order as the error e^H (we recall that $\hat{h} = h/\varepsilon = (N_{mic})^{-(1/d)}$, see Remark 1).

3. Numerical experiment

We consider PDE (1) on the non-convex L-shape domain $\Omega = (-1, 1)^2 \backslash [0, 1] \times [-1, 0]$ where we used the tensor $a(\frac{x}{\varepsilon}) = \frac{64}{9\sqrt{17}}(\sin(2\pi \frac{x_1}{\varepsilon}) + \frac{9}{8})(\cos(2\pi \frac{x_2}{\varepsilon}) + \frac{9}{8}) \cdot I_2$ with coefficients chosen in such a way that the homogenized tensor a^0 matches the unit matrix I_2, see [14, Chap. 1.2]. Then, for Dirichlet boundary conditions $g_D = u^0$, the exact homogenized solution is given by $u^0(r) = r^{\frac{2}{3}} \sin(\frac{2}{3}\vartheta)$ where $r^2 := x^2 + y^2$ and $\vartheta := \tan^{-1}(y/x) \in [0, 2\pi)$. Our adaptive

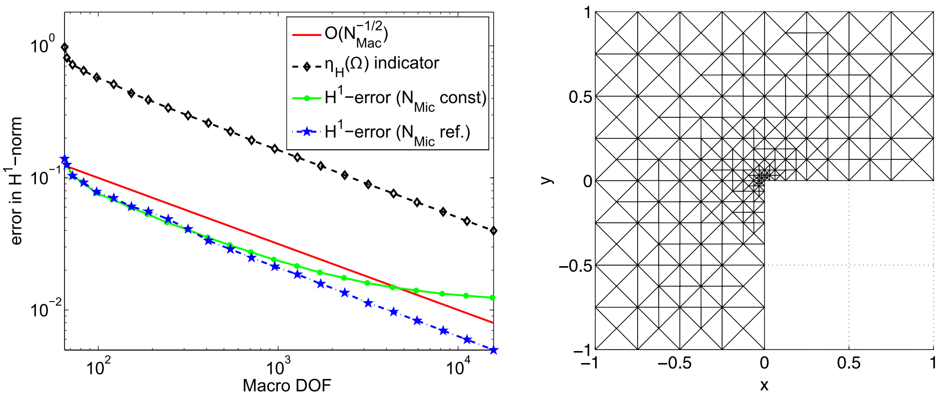

Fig. 1. Errors and indicator estimates in the H^1-norm (left picture) and grid obtained by the adaptive procedure after 9 refinement steps (right picture).

FE-HMM code is based on [6] and a modified version of AFEM@Matlab, see [9]. We choose $\varepsilon = \delta = 10^{-3}$, start with an initial $N_{mic} = 8^2$ and for the 23 refinement steps we used Dörfler's bulk-chasing marking strategy with a parameter $\theta = 0.3$ (see [18, Chap. 4.1], [10]).

Fig. 1 shows the error in the H^1 norm. We choose the size $\hat{h}$ of the micro-problem to be $\hat{h}_K = \sqrt{H_K}$ (see Remark 2). We also plot the value of the indicator $\eta_H(\Omega)$. We see that both the error and the indicators converge to zero with rate $\mathcal{O}(H) = \mathcal{O}(N_{mac}^{-1/2})$, confirming numerically our theoretical estimates. Finally we also plot the error obtained using the same adaptive strategy but with micro-problems and indicators computed on a micro mesh of constant size $\hat{h} = \mathcal{O}(1/10)$. We see that the asymptotic convergence rate is incorrect in this case, showing the importance to perform a simultaneous (macro and micro) mesh refinement. Further numerical experiments and comparisons will be reported in [7].

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