

Differential Geometry/Mathematical Physics

There are no conformal Einstein rescalings of complete pseudo-Riemannian Einstein metrics

Volodymyr Kiosak, Vladimir S. Matveev*

Received 14 May 2009; accepted after revision 25 June 2009

Available online 4 August 2009

Presented by Étienne Ghys

Abstract

Let g be an Einstein metric of indefinite signature such that the conformally-equivalent metric $\psi^{-2}g$ is also Einstein. We show that if the metric g is light-line complete, then the conformal coefficient ψ is constant. If the manifold is closed, the completeness assumption can be omitted (the latter result is due to Mikeš–Radulovich and Kühnel, but our proof is much simpler).

The proof is based on the investigation of the behavior of the function ψ along light-line geodesics: we show that for every light-line geodesic $\gamma(t)$ we have $\psi(\gamma(t)) = \text{const}_1 \cdot t + \text{const}_2$. Since the function ψ cannot vanish, the light-line completeness of the metric implies $\psi = \text{const}_2$.

If the manifold is closed, the function ψ accepts its maximal value ψ_{\max} at a certain point. Then, for every light-line geodesic γ through this point we have $\text{const}_1 = 0$ implying $\psi = \psi_{\max}$ at every point of this geodesic. Repeating the argumentation, we obtain that for every light-line geodesic γ_1 intersecting γ we have $\psi = \psi_{\max}$ at every point of γ_1 as well and so on. Since every two points can be connected by a sequence of light-line geodesics, ψ is constant on the whole manifold. **To cite this article:** V. Kiosak, V.S. Matveev, *C. R. Acad. Sci. Paris, Ser. I 347 (2009)*.

© 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Il n'existe pas de rescaling conformément Einstein d'une métrique d'Einstein pseudo-Riemannienne complète. Soit g une métrique pseudo-riemannienne non définie de type Einstein telle que la métrique conformément équivalente $\psi^{-2}g$ soit aussi d'Einstein. Nous montrons que si la métrique g est lumière-complète, i.e. ses géodésiques isotropes sont complètes, alors le coefficient ψ est constant. Si la variété est fermée, l'hypothèse de complétude peut être omise (ce dernier résultat est dû à Mikeš–Radulovich et Kühnel, mais notre démonstration est plus courte).

La démonstration est basée sur l'étude du comportement de la fonction ψ le long des géodésiques de type lumière. Si $\gamma(t)$ est une telle géodésique, alors : $\psi(\gamma(t)) = \text{const}_1 \cdot t + \text{const}_2$. Comme la fonction ψ est non-nulle, la lumière-complétude implique $\psi = \text{const}_2$.

Si la variété est fermée, la fonction ψ prend sa valeur maximale ψ_{\max} en un certain point. Donc, pour toute géodésique de type lumière γ passant par ce point, on a $\text{const}_1 = 0$, ce qui implique que $\psi = \psi_{\max}$ en tout point de cette géodésique. En répétant cet argument, on obtient que pour toute géodésique γ_1 de type lumière coupant γ , $\psi = \psi_{\max}$ en tout point de γ_1 , et ainsi de suite. On en déduit ψ est constante sur la variété entière, car deux points quelconques peuvent être joints par une suite de géodésiques de type lumière. **Pour citer cet article :** V. Kiosak, V.S. Matveev, *C. R. Acad. Sci. Paris, Ser. I 347 (2009)*.

* Institute of Mathematics, FSU Jena, 07737 Jena, Germany.
E-mail address: vladimir.matveev@uni-jena.de (V.S. Matveev).

Theorem 1. *Let g be a light-line-complete pseudo-Riemannian Einstein metric of indefinite signature (i.e., for no constant c the metric $c \cdot g$ is Riemannian) on a connected ($n > 2$)-dimensional manifold M . Assume that for the nowhere vanishing function ψ the metric $\psi^{-2}g$ is also Einstein. Then, ψ is a constant.*

Remark 1. Theorem 1 fails for Riemannian metrics (even if we replace light-line completeness by usual completeness) – Möbius transformations of the standard round sphere and the stereographic map of the punctured sphere to the Euclidean space are conformal nonhomothetic mappings. One can construct other examples on warped Riemannian manifolds, see [6, Theorem 21].

Remark 2. By Theorem 1, *light-line complete pseudo-Riemannian Einstein metrics of indefinite signature do not admit nonhomothetic conformal complete vector fields.* The Riemannian version of this result is due to Yano and Nagano [11]. Moreover, the assumption that the metric is Einstein can be omitted (by the price of considering only essential conformal vector fields): as it was proved by D. Alekseevskii [1], J. Ferrand [3] and R. Schoen [10], a Riemannian manifold admitting an essential complete vector field is conformally equivalent to the round sphere or the Euclidean space. It is still not known whether the last statement (sometimes called Lichnerowicz–Obata conjecture) can be extended to the pseudo-Riemannian case, see [8] for a counterexample in the C^1 -smooth category, and [4,5] for a good survey on this topic.

Remark 3. In the 4-dimensional Lorenz case, Theorem 1 was known in folklore: more precisely, conformal Einstein rescalings of 4-dimensional Einstein metrics were described by Brinkmann [2], see also [7, Corollary 2.10]. The list of all such metrics and their conformal Einstein rescalings is pretty simple and one can directly verify our Theorem 1 by calculations.

Remark 4. A partial case of Theorem 1 is [7, Theorem 2.2], in which it is assumed that both metrics are complete. This extra-assumption is very natural in the context of [7] since the paper is dedicated to the classification of conformal vector fields; moreover, Theorem 2.2 is not the main result of the paper. It is not clear whether in the proof of [7, Theorem 2.2] the assumption that the second metric is complete could be omitted.

Proof of Theorem 1. It is well known (see for example [2, Eq. (2.21)] or [6, Lemma 1]) that the Ricci curvatures R_{ij} and \bar{R}_{ij} of two conformally equivalent metrics g and $\bar{g} = \psi^{-2}g = e^{-2\phi}g$ are related by

$$\bar{R}_{ij} = R_{ij} + (\Delta\phi - (n-2)\|\nabla\phi\|^2)g_{ij} + \frac{n-2}{\psi}\nabla_i\nabla_j\psi. \quad (1)$$

Consider a light-line geodesic $\gamma(t)$ of the metric g . Since the metric g is light-line-complete, $\gamma(t)$ is defined on the whole \mathbb{R} . “Light-line” means that $g(\dot{\gamma}(t), \dot{\gamma}(t)) = g_{ij}\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0$, where $\dot{\gamma}$ is the velocity vector of γ (it is well known that if this property is fulfilled in one point then it is fulfilled at every point of the geodesic).

Now contract (1) with $\dot{\gamma}^i\dot{\gamma}^j$. Since the metrics are Einstein and conformally equivalent, \bar{R}_{ij} , R_{ij} and g_{ij} are proportional to g_{ij} , and therefore the only term which does not vanish is $\dot{\gamma}^i\dot{\gamma}^j\frac{n-2}{\psi}\nabla_i\nabla_j\psi$. Thus, $\dot{\gamma}^i\dot{\gamma}^j\nabla_i\nabla_j\psi = 0$.

Clearly, at every point of the geodesic we have $\dot{\gamma}^i\dot{\gamma}^j\nabla_i\nabla_j\psi = \frac{d^2}{dt^2}\psi(\gamma(t))$. Thus, $\frac{d^2}{dt^2}\psi(\gamma(t)) = 0$ implying $\psi(\gamma(t)) = \text{const}_1 \cdot t + \text{const}$. Since by assumptions the function ψ is defined on the whole \mathbb{R} and is equal to zero at no point, we have $\text{const}_1 = 0$ implying $\psi \equiv \text{const}$ along every light-line geodesic.

Now, every two points of a connected manifold can be connected by a finite sequence of light-line geodesics. Indeed, consider \mathbb{R}^n with the standard pseudo-Euclidean metric g_0 of the same signature $(r, n-r)$, $1 \leq r < n$ as the metric g . The union of all light-line geodesics passing through points a (resp. b) are the standard cones $C_a := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1 - a_1)^2 + \dots + (x_r - a_r)^2 - (x_{r+1} - a_{r+1})^2 - \dots - (x_n - a_n)^2 = 0\}$ and, resp., $C_b := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1 - b_1)^2 + \dots + (x_r - b_r)^2 - (x_{r+1} - b_{r+1})^2 - \dots - (x_n - b_n)^2 = 0\}$. These two cones always have points of transversal intersection. Thus, two arbitrary points of \mathbb{R}^n can be connected by a sequence of two light-line geodesics of g_0 . Since the restriction of the metric g to a small neighborhood $U \subseteq M^n$ can be viewed

as a small perturbation of the metric g_0 in \mathbb{R}^n , two points in U can be connected by a sequence of two light-line geodesics. Then, the set of points of M that can be connected with a fixed point $p \in M^n$ by a finite sequence of light-line geodesics is open and closed implying it coincides with M .

Since every two points of M can be connected by a sequence of light-line geodesics, and since as we proved above the function ψ is constant along every light-line geodesic, we have that ψ is constant on the whole manifold as we claimed. \square

Theorem 2. *Let g be a pseudo-Riemannian Einstein metric of indefinite signature on a connected closed (i.e., compact with no boundary) $(n > 2)$ -dimensional manifold M . Assume that for the nowhere vanishing function ψ the metric $\psi^{-2}g$ is also Einstein. Then, ψ is a constant.*

Remark 5. Theorem 2 is not new and is in [9, Theorem 5]. Moreover, Wolfgang Kühnel explained us how one can obtain the proof combining the results of PhD thesis of Kerckhove, Eq. (1) due to [2], and also [7, Proposition 3.8(1)]. Our proof of Theorem 2 is much easier than the proofs of Mikeš–Radulovich and Kühnel. Actually, the initial version of our paper did not contain Theorem 2 at all, but after J. Mikeš sent us his paper we immediately saw that the proof of their Theorem 5 can be essentially simplified by using the trick from the proof of our Theorem 1.

Proof of Theorem 2. Since M is closed, there exists $p_0 \in M$ such that the value of ψ is maximal (we denote this value by ψ_{max}). We take a light-line geodesic γ such that $\gamma(0) = p_0$. As we explained in the proof of Theorem 1, the function $\psi(\gamma(t))$ is equal to $\text{const} \cdot t + \psi_{max}$. Since the value of ϕ at the point p_0 is maximal, $\text{const} = 0$ implying $\psi(\gamma(t)) \equiv \psi_{max}$. Then, for every point p_1 of geodesic γ the value of ψ is maximal. We can therefore repeat the argumentation and show that for every light-line geodesic γ_1 such that $\gamma_1(0) = p_1$ we have $\psi(\gamma_1(t)) \equiv \psi_{max}$ and so on. Since every two points of M can be connected by a sequence of light-line geodesics, we have that ψ is constant on the whole manifold. \square

Acknowledgements

We thank W. Kühnel and H.-B. Rademacher for sending us the preliminary version of their paper [7] and for useful discussions, and J. Mikeš for sending us his paper [9]. When we obtained the proof, we asked all experts we know whether the proof is new, and are grateful to those who answered, in particular to M. Eastwood, Ch. Frances, R. Gover, G. Hall, F. Leitner, and P. Nurowski. We thank Konrad Schöbel and Ghani Zeghib for helping us with the translation of the title and the abstract to French. Both authors were partially supported by Deutsche Forschungsgemeinschaft (Priority Program 1154 – Global Differential Geometry), and by FSU Jena.

References

- [1] D.V. Alekseevskii, Groups of conformal transformations of Riemannian spaces, *Mat. Sb.* 89 (131) (1972) (in Russian); Engl. transl.: *Math. USSR Sb.* 18 (1972) 285–301.
- [2] H.W. Brinkmann, Einstein spaces which are mapped conformally on each other, *Math. Ann.* 94 (1925) 119–145.
- [3] J. Ferrand, The action of conformal transformations on a Riemannian manifold, *Math. Ann.* 304 (1996) 277–291.
- [4] Ch. Frances, Essential Conformal Structures in Riemannian and Lorentzian Geometry. Recent Developments in Pseudo-Riemannian Geometry, *ESI Lect. Math. Phys., Eur. Math. Soc., Zürich*, 2008, pp. 231–260.
- [5] Ch. Frances, Sur le groupe d’automorphismes des géométries paraboliques de rang 1, *Ann. Sci. École Norm. Sup.* (4) 40 (5) (2007) 741–764, English translation available on <http://www.math.u-psud.fr/~frances/cartan-english6.pdf>.
- [6] W. Kühnel, Conformal transformations between Einstein spaces, in: *Conformal Geometry*, Bonn, 1985/1986, in: *Aspects Math.*, vol. E12, Vieweg, Braunschweig, 1988, pp. 105–146.
- [7] W. Kühnel, H.-B. Rademacher, Einstein spaces with a conformal group, *Results in Math.*, in press.
- [8] F. Leitner, Twistor spinors with zero on Lorentzian 5-space, *Comm. Math. Phys.* 275 (3) (2007) 587–605.
- [9] J. Mikeš, Zh. Radulovich, On concircular and torse-forming vector fields “in the large”, *Math. Montisnigri* 4 (1995) 43–54 (in Russian).
- [10] R. Schoen, On the conformal and CR automorphism groups, *Geom. Funct. Anal.* 5 (2) (1995) 464–481.
- [11] K. Yano, T. Nagano, Einstein spaces admitting a one-parameter group of conformal transformations, *Ann. Math.* (2) 69 (1959) 451–461.