Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 347 (2009) 947–952
COMPTES RENDUS MATHEMATIQUE

Dynamical Systems

Affability of Euclidean tilings ☆

Fernando Alcalde Cuesta [a], Pablo González Sequeiros [a], Álvaro Lozano Rojo [b]

[a] *Dpto. Xeometría e Topoloxía, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain*
[b] *Dpto. Matemáticas, Universidad del País Vasco-Euskal Herriko Unibertsitatea, 48940 Leioa, Spain*

Received 11 December 2007; accepted after revision 17 June 2009

Available online 2 July 2009

Presented by Étienne Ghys

Abstract

We prove that every minimal equivalence relation on a Cantor set arising from the continuous hull of an aperiodic and repetitive Euclidean tiling is affable. ***To cite this article: F. Alcalde Cuesta et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Affabilité des pavages euclidiens. Nous prouvons que toute relation d'équivalence définie sur l'ensemble de Cantor par l'enveloppe d'un pavage euclidien apériodique et répétitif est affable. ***Pour citer cet article : F. Alcalde Cuesta et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Un *pavage euclidien* est une décomposition de $\mathbb{R}^m$ en polyèdres, appelés *pavés*, qui s'intersèquent face à face. Ces pavés sont obtenus par translation à partir d'un ensemble fini de *protopavés*. Un pavage est *apériodique* s'il n'est invariant par aucune translation et il est *répétitif* si pour chaque motif M, il existe $R > 0$ tel que toute boule de rayon R contient une copie par translation de M. Soit $\mathfrak{T}(\mathcal{P})$ l'ensemble de pavages $\mathcal{T}$ obtenus à partir d'un ensemble fini de protopavés $\mathcal{P}$. On peut munir $\mathfrak{T}(\mathcal{P})$ de la *topologie de Gromov–Hausdorff* [2,3] engendrée par les voisinages ouverts de $\mathcal{T}$ qui sont formés des pavages $\mathcal{T}'$ tels que $R(\mathcal{T}+v, \mathcal{T}'+v') > r$ où $\|v\| < \varepsilon$, $\|v'\| < \varepsilon'$ et $R(\mathcal{T}, \mathcal{T}')$ est le plus grand rayon $R > 0$ tel que $\mathcal{T}$ et $\mathcal{T}'$ coïncident sur la boule $B(0, R)$. L'ensemble $\mathfrak{T}(\mathcal{P})$ devient ainsi un espace compact métrisable, feuilleté par les orbites $L_{\mathcal{T}}$ de l'action naturelle de $\mathbb{R}^m$. Le choix d'un point base dans chaque protopavé détermine un ensemble de Delone $D_{\mathcal{T}}$ pour tout $\mathcal{T} \in \mathfrak{T}(\mathcal{P})$. Alors $\Sigma = \{\mathcal{T} \in \mathfrak{T}(\mathcal{P}) / 0 \in D_{\mathcal{T}}\}$ est un fermé totalement disconnexe qui rencontre toute feuille. Pour tout pavage apériodique et répétitif $\mathcal{T} \in \mathfrak{T}(\mathcal{P})$, on appelle *enveloppe de* $\mathcal{T}$ la fermeture de $L_{\mathcal{T}}$. Dans le cas euclidien, tout pavage de $\mathfrak{X} = \overline{L}_{\mathcal{T}}$ est aussi apériodique et répétitif et

☆ This work was supported by the Spanish Ministry of Education and Science (Research Projects MTM2004-08214 and MTM2007-66262), the University of the Basque Country (R. Project EHU 06/05), and the Spanish Network of Topology.

E-mail addresses: fernando.alcalde@usc.es (F. Alcalde Cuesta), pablo.gonzalez.sequeiros@usc.es (P. González Sequeiros), alvaro.lozano@ehu.es (Á. Lozano Rojo).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.06.011

donc $X = \Sigma \cap \mathfrak{X}$ est homéomorphe à l'ensemble de Cantor. Le feuilletage induit $\mathcal{F} = \{L_{\mathcal{T}}\}_{\mathcal{T} \in \mathfrak{X}}$ s'identifie alors avec la relation d'équivalence β-discrète $\mathcal{R} = \{(\mathcal{T}, \mathcal{T} - v) \in X \times X / v \in D_{\mathcal{T}}\}$ et sa dynamique transverse avec la classe d'isomorphisme stable de $\mathcal{R}$. Deux relations d'équivalence β-discrètes $\mathcal{R}$ et $\mathcal{R}'$ définies sur des espaces polonais X and X' sont *stablement isomorphes* (resp. *stablement orbitalement équivalentes*) s'il existe des ouverts Y dans X et Y' dans X' qui rencontrent toute classe d'équivalence de $\mathcal{R}$ et $\mathcal{R}'$ tels que les relations d'équivalence induites $\mathcal{R}|_Y$ et $\mathcal{R}'|_{Y'}$ sont isomorphes (resp. orbitalement équivalentes). Dans le cas mesurable, Y et Y' doivent être des boréliens qui rencontrent presque toute classe d'équivalence.

Une famille de 32 protopavés apériodiques (6 à isométrie près) a été décrite dans [13]. La construction d'un pavage de Robinson répétitif se fait alors à partir de choix succésifs de couples 00, 01, 11 et 10. Par ailleurs, d'après une idée de R.M. Robinson [8], tout pavage de Penrose en flèches et cerfs volants est codé par une suite $\{x_n\}_{n \in \mathbb{N}}$ de 0 et de 1 telle que $x_n = 1 \Rightarrow x_{n+1} = 0$. Dans les deux cas, si on modifie les premiers termes d'une suite, l'origine est translaté d'un pavé à un autre. La dynamique transverse des pavages de Robinson et de Penrose est ainsi représentée (d'une manière mesurable) par la somme de 1 dans les entiers dyadiques et par l'automate de Fibonacci respectivement. Le but de cette Note est de prouver que la dynamique transverse de $\mathfrak{X}$ est représentée (à équivalence orbitale près) par la relation cofinale sur l'espace des chemins infinis d'un diagramme de Bratteli. Une telle relation est orbitalement équivalente à une action minimale de $\mathbb{Z}$ sur l'ensemble de Cantor [4]. Autrement dit, la relation d'équivalence $\mathcal{R}$ est *affable* au sens de [4]. Nous généralisons ainsi un résultat de [10], valable pour certains pavages de substitution, mais qui ne s'applique pas aux exemples précédents. Après la soumission de ce travail, un théorème d'affabilité pour les $\mathbb{Z}^m$-systèmes minimaux sur l'ensemble de Cantor a été annoncé dans [7] (complétant le cas $m = 2$ traité par les mêmes auteurs [5]). Grâce au choix de la transversale Σ, on peut ramener notre résultat à celui de [7]. Réciproquement, par un procédé standard de suspension, le résultat de [7] est une conséquence de notre résultat. Notre démarche reste cependant valable pour des laminations auxquelles le résultat de [7] ne s'applique pas [1,3,9]. La preuve se fait en quatre étapes :

i) Le but de la première étape est de construire une sous-relation d'équivalence affable $\mathcal{R}_\infty \subset \mathcal{R}$. Nous utiliserons la démarche d'*inflation* ou *zooming* décrite dans [2]. Nous aurons ainsi une suite de décompositions $\mathcal{B}^{(n)}$ de $\mathfrak{X}$ en un nombre fini de sous-ensembles compacts feuilletés en produit. Alors $\mathcal{R}_\infty$ est la limite inductive d'une suite de relations d'équivalence étales compactes $\mathcal{R}_n$ définies par ces décompositions.
ii) Dans la deuxième étape, nous commencerons par définir le *bord de* $\mathcal{R}_\infty$. Il s'agit d'un fermé d'intérieur vide $\partial\mathcal{R}_\infty$ dont chaque $\mathcal{R}$-classe du saturé se décompose en au moins deux $\mathcal{R}_\infty$-classes. Nous exhiberons ensuite une propriété importante du bord dans le cas où le pavage est euclidien de type fini, à savoir chaque $\mathcal{R}$-classe se décompose en un nombre fini de $\mathcal{R}_\infty$-classes et la sous-relation $\mathcal{R}_\infty$ reste minimale.
iii) La troisième étape consistera à prouver que $\partial\mathcal{R}_\infty$ est $\mathcal{R}$-*fin* [4], i.e. $\mu(\partial\mathcal{R}_\infty) = 0$ pour toute mesure de probabilité $\mathcal{R}$-invariante μ. Nous nous servirons d'un lemme de type Rohlin que nous déduirons du type de croissance des feuilles de $\mathfrak{X}$. Pour démontrer ce lemme, nous utiliserons une méthode analogue à celle employée par C. Series dans [14].
iv) Pour compléter la preuve, nous nous inspirerons de la démarche utilisée dans [10]. Mais grâce aux étapes précédentes, nous n'aurons pas besoin de nous limiter aux pavages de substitution planaires. Ainsi, si toute $\mathcal{R}$-classe se découpe en au plus deux $\mathcal{R}_\infty$-classes, nous décomposerons $\mathcal{R}$ en une famille dénombrable de bissections et nous concluirons de proche en proche par application du théorème 4.18 de [4]. Enfin, en utilisant les propriétés des décompositions $\mathcal{B}^{(n)}$, nous pourrons réduire le cas général au cas précédent.

1. Introduction and preliminaries

An *Euclidean tiling* is a partition of $\mathbb{R}^m$ into *tiles*, which are polyhedra touching face to face. These tiles are obtained by translation from a finite set of *prototiles*. A tiling is said to be *aperiodic* if it has no translation symmetries. It is said to be *repetitive* if for any patch M, there exists $R > 0$ such that any ball of radius R contains a translated copy of M. Let $\mathfrak{T}(\mathcal{P})$ be the set of tilings $\mathcal{T}$ obtained from a finite set of prototiles $\mathcal{P}$. Then it is possible to endow $\mathfrak{T}(\mathcal{P})$ with the *Gromov–Hausdorff topology* [2,3] generated by the basic neighbourhoods

$$U^r_{\varepsilon,\varepsilon'} = \left\{\mathcal{T}' \in \mathfrak{T}(\mathcal{P}) / \exists v, v' \in \mathbb{R}^m\colon\ \|v\| < \varepsilon, \|v'\| < \varepsilon', R(\mathcal{T} + v, \mathcal{T}' + v') > r\right\}$$

where $R(\mathcal{T},\mathcal{T}')$ is the supremum of radii $R>0$ such that $\mathcal{T}$ and $\mathcal{T}'$ coincide on the ball $B(0,R)$. Thus $\mathfrak{T}(\mathcal{P})$ becomes a compact metrizable space which is foliated by the orbits $L_{\mathcal{T}}$ of the natural action of $\mathbb{R}^m$. For each $\mathcal{T}\in\mathfrak{T}(\mathcal{P})$, let $D_{\mathcal{T}}$ be the Delone set determined by the choice of base points in the prototiles. Now $\Sigma=\{\mathcal{T}\in\mathfrak{T}(\mathcal{P})/0\in D_{\mathcal{T}}\}$ is a totally disconnected closed subspace which meets all the leaves. For any aperiodic and repetitive tiling $\mathcal{T}\in\mathfrak{T}(\mathcal{P})$, the *continuous hull* of $\mathcal{T}$ is the closure of $L_{\mathcal{T}}$. In the Euclidean case, any tiling in $\mathfrak{X}=\overline{L}_{\mathcal{T}}$ is also aperiodic and repetitive and therefore $X=\Sigma\cap\mathfrak{X}$ is homeomorphic to the Cantor set. Now we can identify the induced foliation $\mathcal{F}=\{L_{\mathcal{T}}\}_{\mathcal{T}\in\mathfrak{X}}$ with the étale equivalence relation (EER) $\mathcal{R}=\{(\mathcal{T},\mathcal{T}-v)\in X\times X/v\in D_{\mathcal{T}}\}$ and its transverse dynamics with the stable isomorphism class of $\mathcal{R}$. Two EERs $\mathcal{R}$ and $\mathcal{R}'$ on polish spaces X and X' are *stably isomorphic* (resp. *stably orbit equivalent*) if there exists open subsets Y of X and Y' of X' meeting every equivalence class of $\mathcal{R}$ and $\mathcal{R}'$ such that the induced equivalence relations $\mathcal{R}|_Y$ and $\mathcal{R}'|_{Y'}$ are isomorphic (resp. orbit equivalent). In the measurable case, Y and Y' must be borelian sets meeting a.e. equivalence class.

A set of 32 aperiodic tiles (6 up to isometries of the plane) was described in [13]. Any repetitive Robinson tiling depends on successive choices of pairs 00, 01, 11, 10 made in the course of its construction. On the other hand, according to an idea by R.M. Robinson [8], any Penrose tiling with kites and darts is encoded by a sequence $\{x_n\}_{n\in\mathbb{N}}$ of 0's and 1's such that $x_n=1\Rightarrow x_{n+1}=0$. In both cases, if we replace the first terms in a sequence, then the origin is translated from a tile to another one. The transverse dynamics of Robinson and Penrose tilings is thus represented (in a measurable way) by the 2-adic odometer and the Fibonacci automaton respectively. The aim of this Note is to show that the transverse dynamics of $\mathfrak{X}$ is represented (up to orbital equivalence) by the tail equivalence relation on the infinite path space of a Bratteli diagram. Such an equivalence relation is orbit equivalent to a minimal $\mathbb{Z}$-action on the Cantor set [4]. In other words, the equivalence relation $\mathcal{R}$ is *affable* in the sense of [4]. This generalizes a result of H. Matui [10] for a class of substitution tilings (which does not include the examples above). After the submission of this work, an affability theorem for Cantor minimal $\mathbb{Z}^m$-systems has been announced in [7] (extending an earlier result for $m=2$ [5]). By using the transversal Σ, our result can be reduced to the result of [7] up to orbital equivalence (see Remark 2.33 of [2]). Reciprocally, by a standard suspension method, this result can be deduced from ours. However, our proof is valid for laminations to which the theorem of [7] does not apply [1,3,9].

Let $\mathcal{R}$ be an EER on a locally compact space X. Following [4], we say that $\mathcal{R}$ is a *compact étale equivalence relation* (CEER) if $\mathcal{R}-\Delta$ is a compact subset of $X\times X$ (where Δ is the diagonal of $X\times X$). This means that $\mathcal{R}$ is proper and trivial out of a compact set.

Definition 1.1. (See [4].) An equivalence relation $\mathcal{R}$ on a totally disconnected space X is *affable* if there exists an increasing sequence of CEERs $\mathcal{R}_n$ such that $\mathcal{R}=\bigcup_{n\in\mathbb{N}}\mathcal{R}_n$. The inductive limit topology turns $\mathcal{R}$ into an EER and we say that $\mathcal{R}=\varinjlim\mathcal{R}_n$ is *approximately finite* (or AF).

An example is the cofinal or tail equivalence relation on the infinite path space of a particular type of oriented graphs, called *Bratteli diagrams* [4], whose vertices are stacked on levels and whose edges join consecutive levels. They actually are the only examples of AF equivalence relation [4]. Indeed, for any AF equivalence relation $\mathcal{R}$ on a compact space X, there exists a standard Bratteli diagram (V,E) such that the tail equivalence relation on the infinite path space $X_{(V,E)}$ is isomorphic to $\mathcal{R}$. As in the case of the 2-adic odometer and the Fibonacci automaton, the tail equivalence relation on the infinite path space of a *ordered Bratteli diagram* (having a linear order on each set of edges with the same endpoint) is essentially isomorphic to a Cantor $\mathbb{Z}$-system [4].

2. Inflation

In order to describe the transverse dynamics of $\mathfrak{X}$, we need the *inflation* or *zooming process* developed in [2]. By definition $\mathfrak{X}$ admits a *flow box decomposition* $\mathcal{B}=\{B_i\}_{i=1}^k$ consisting of closed flow boxes $\varphi_i:B_i\to P_i\times X_i$ such that $\mathfrak{X}=\bigcup_{i=1}^k B_i$ and $\mathring{B}_i\cap\mathring{B}_j=\emptyset$ if $i\neq j$. The set $\partial_v B_i=\varphi_i^{-1}(\partial P_i\times X_i)$ is the *vertical boundary* of B_i. The flow box decomposition is *well-adapted* if each plaque P_i is a $\mathcal{P}$-patch and the axis $\bigsqcup_{i=1}^k X_i$ is a clopen subset of X.

Theorem 2.1. *(See [2].) Let $\mathfrak{X}$ be the continuous hull of an aperiodic and repetitive Euclidean tiling in $\mathfrak{T}(\mathcal{P})$. Then, for any well-adapted flow box decomposition $\mathcal{B}$ of $\mathfrak{X}$, there exists another well-adapted flow box decomposition $\mathcal{B}'$* zoomed out of *$\mathcal{B}$, i.e.*

i) *for each tiling $\mathcal{T}$ in a box $B \in \mathcal{B}$ and in a box $B' \in \mathcal{B}'$, the transversal of B' through $\mathcal{T}$ is contained in the corresponding transversal of B*;
ii) *the vertical boundary of boxes of $\mathcal{B}'$ is contained in the vertical boundary of boxes of $\mathcal{B}$*;
iii) *for each box $B' \in \mathcal{B}'$, there exists a box $B \in \mathcal{B}$ such that $B \cap B' \neq \emptyset$ and $B \cap \partial_v B' = \emptyset$.*

By recurrence, we obtain a sequence of well-adapted flow box decompositions $\mathcal{B}^{(n)}$ such that $\mathcal{B}^{(0)} = \mathcal{B}$, $\mathcal{B}^{(n+1)}$ is zoomed out of $\mathcal{B}^{(n)}$ and $\mathcal{B}^{(n+1)}$ defines a finite set $\mathcal{P}^{(n+1)}$ of $\mathcal{P}^{(n)}$-patches (containing at least a $\mathcal{P}^{(n)}$-tile in its interior) and a tiling in $\mathfrak{T}(\mathcal{P}^{(n+1)})$ of each leaf of $\mathfrak{X}$. Let $X^{(n)}$ be the axis of $\mathcal{B}^{(n)}$. Given a increasing sequence of integers k_n, we can also suppose that each Delone set $D_{\mathcal{T}}^{(n)} = L_{\mathcal{T}} \cap X^{(n)}$ is k_n-separated. Such a sequence defines a increasing sequence of CEERs $\mathcal{R}_n$ on X. Indeed, for each $n \in \mathbb{N}$, the class $\mathcal{R}_n[\mathcal{T}]$ is the *discrete plaque* $\check{P} = P \cap X$ determined by the plaque $P \in \mathcal{P}^{(n)}$ of $\mathcal{B}^{(n)}$ through $\mathcal{T}$. Therefore $\mathcal{R}_\infty = \varinjlim \mathcal{R}_n$ is an open AF equivalence subrelation of $\mathcal{R}$.

3. Boundary

For any tiling $\mathcal{T} \in X$, let $\partial\mathcal{R}_n[\mathcal{T}]$ be the set of tilings $\mathcal{T}' = \mathcal{T} - v$ such that v is the base point of a tile in P meeting ∂P. We define the *boundary of* $\mathcal{R}_n$ as the clopen set $\partial\mathcal{R}_n = \bigcup_{\mathcal{T} \in X^{(n)}} \partial\mathcal{R}_n[\mathcal{T}] = \bigcup_{B \in \mathcal{B}^{(n)}} \partial_v \check{B}$ where $\partial_v \check{B}$ is the vertical boundary of the *discrete flow box* $\check{B} = B \cap X$. Now $\partial\mathcal{R}_\infty = \bigcap_{n \in \mathbb{N}} \partial\mathcal{R}_n$ is a meager closed subset of X, which we shall call the *boundary of* $\mathcal{R}_\infty$. If $\mathcal{T} \in \partial\mathcal{R}_\infty$, then $\mathcal{R}[\mathcal{T}]$ separates into several $\mathcal{R}_\infty$-classes. By construction, each one of these $\mathcal{R}_\infty$-classes contains an increasing sequence of discrete plaques defined by an increasing sequence of $\mathcal{P}^{(n)}$-tiles. In the Euclidean case, the union of these tiles contains a conical region. Any open subset A of X meets this region because $A \cap L_{\mathcal{T}}$ is a Delone set quasi-isometric to $L_{\mathcal{T}}$. This implies the following result:

Lemma 3.1. *The AF equivalence relation $\mathcal{R}_\infty$ is minimal. Moreover, any $\mathcal{R}$-class splits into a finite number of $\mathcal{R}_\infty$-classes.*

Our aim in the next section is to prove another important property of $\partial\mathcal{R}_\infty$:

Lemma 3.2. *The set $\partial\mathcal{R}_\infty$ is $\mathcal{R}$-thin, i.e. $\mu(\partial\mathcal{R}_\infty) = 0$ for every $\mathcal{R}$-invariant probability measure μ.*

4. Isoperimetric inequalities and Rohlin lemma for Euclidean tilings

For simplicity, we start with the assumption that $m = 2$. In this case, any convex polyhedron V containing a ball of radius $k > 0$ satisfies the inequality $length(\partial V)/area(V) < \frac{2\pi}{k}$. Thus, for any sequence $x_n \in D_{\mathcal{T}}^{(n)}$, there exists a sequence of Voronoi cells V_n such that $\lim_{n\to\infty} length(\partial V_n)/area(V_n) = \lim_{n\to\infty} 2\pi/k_n = 0$. Now a suitable choice of k_n (namely $k_n > l_{n-1}^2 + 2\pi n l_{n-1}$ where l_{n-1} is the maximum length of the boundaries of $\mathcal{P}^{(n-1)}$-tiles) allows us to construct a sequence of box decompositions $\mathcal{B}^{(n)}$ such that the isoperimetric ratio of any sequence of plaques $P_n \in \mathcal{P}^{(n)}$ converges uniformly to 0, i.e. $length(\partial P_n)/area(P_n) < \frac{1}{n}$, $\forall n \in \mathbb{N}$. In fact, we can discretize this construction by replacing P_n by $\check{P}_n$ and choosing $k_n > l_{n-1}^2 + 2\pi n a_{n-1} l_{n-1}$ where a_{n-1} is the maximum area of the $\mathcal{P}^{(n-1)}$-tiles. Here, we write $area(P)$ and $length(\partial P)$ for the usual volume of P and ∂P, as well as for the discrete counterparts $area(P) = \#\check{P}$ and $length(\partial P) = \#\partial\check{P}$. As an elementary consequence of the remark above, we obtain:

Lemma 4.1. *For each $n \in \mathbb{N}$ and each $B \in \mathcal{B}^{(n)}$, we have $\mu(\partial_v \check{B}) < \frac{1}{n}\mu(\check{B})$. Therefore $\mu(\partial\mathcal{R}_n) < \frac{1}{n}$ and so $\mu(\partial\mathcal{R}_\infty) = \lim_{n\to\infty} \mu(\partial\mathcal{R}_n) = 0$.*

In higher dimensions, we need to compare the area of an inscribed simplex with the length of a circumscribed cube to obtain a good isoperimetric inequality. However, we will give another proof which is valid in a more general setting. C. Series has proved in [14] that any foliation with polynomial growth is hyperfinite. To prove Lemma 3.2, we will use the same method, analogous to the usual methods to demonstrate the Rohlin lemma. All the boxes considered shall be discretized, but we will use the notation B for simplicity. We will say that B is a *flow box of axis C and width* $2n$ if B contains a transversal C such that each plaque P through $\mathcal{T} \in C$ is equal to the ball $\bar{B}_{\mathcal{R}[\mathcal{T}]}(\mathcal{T}, n) = \{\mathcal{T}' \in \mathcal{R}[\mathcal{T}] / d(\mathcal{T}, \mathcal{T}') \leqslant n\}$ where $d(\mathcal{T}, \mathcal{T}')$ the length of the shortest path of $\mathcal{P}$-tiles between the origins of $\mathcal{T}$ and $\mathcal{T}'$. We will write $v_{\mathcal{T}}(n) = \#\bar{B}_{\mathcal{R}[\mathcal{T}]}(\mathcal{T}, n)$ and $\partial^r P = \{\mathcal{T}' \in P / \exists \mathcal{T}'' \notin P\colon d(\mathcal{T}', \mathcal{T}'') \leqslant r\}$. The union $B = \bigsqcup_{i=1}^k B_i$ of a family of disjoint flow boxes $B_1, \ldots, B_k$ of axis $C_1, \ldots, C_k$ and width $2n$ will be called a *stack of axis* $C = \bigsqcup_{i=1}^k C_i$ *and width* $2n$. We start with a global version of Jenkins's result used in [14]:

Lemma 4.2. *There are an increasing sequence of locally constant functions* $n_q : X \to \mathbb{N}$ *and a constant* $M > 0$ *such that* $v_{\mathcal{T}}(2n_q(\mathcal{T})) \leqslant M v_{\mathcal{T}}(n_q(\mathcal{T}))$ *and* $\lim_{q\to\infty} v_{\mathcal{T}}(n_q(\mathcal{T}) - r)/v_{\mathcal{T}}(n_q(\mathcal{T})) = 1$ *for each* $\mathcal{T} \in X$ *and each* $r \geqslant 1$.

Proof. For each $\mathcal{T} \in X$, the function $v_{\mathcal{T}}$ is dominated by some polynomial function $p(n) = n^d$ because $v_{\mathcal{T}}$ has polynomial growth. If M denotes the ratio $p(2n)/p(n) = 2^d$, then $\liminf_{n\to\infty} v_{\mathcal{T}}(2n)/v_{\mathcal{T}}(n) \leqslant M$. On the other hand, the subexponential growth of $v_{\mathcal{T}}$ says that any sequence of balls contains a Følner subsequence for which the ratio $v_{\mathcal{T}}(n - r)/v_{\mathcal{T}}(n)$ converges to 1 for all $r \geqslant 1$. To conclude, it is enough to notice that the volume function $v : (\mathcal{T}, n) \in X \times \mathbb{N} \mapsto v_{\mathcal{T}}(n) \in \mathbb{N}$ is continuous. $\square$

By the standard techniques applied in [14], we have:

Lemma 4.3. *Let* A *be a clopen subset of* X *such that* $v_{\mathcal{T}}(2n) \leqslant M v_{\mathcal{T}}(n)$ *for every* $\mathcal{T} \in A$. *For each* $\delta > 0$, *there is a flow box* B *of axis* $C \subset A$ *and width* $\leqslant 2n$ *such that* $\mu(B) > \frac{1}{M}\mu(A) - \delta$. *If we suppose that* $v_{\mathcal{T}}(n-r) < (1-\varepsilon)v_{\mathcal{T}}(n)$ *for any* $\mathcal{T} \in C$, *then* $\mu(\partial^r B) < \varepsilon\mu(B)$.

Lemma 4.4. *Let* A *be a clopen subset of* X. *For each* $r \in \mathbb{N}$ *and each pair* $\delta, \varepsilon > 0$, *there are* $n \geqslant 0$ *and a stack* B *of axis* $C \subset A$ *and width* $\leqslant 2n$ *such that* $\mu(B) > \frac{1}{M}\mu(A) - \delta$ *and* $\mu(\partial^r B) < \varepsilon\mu(B)$.

Lemma 4.5. *For each* $r \in \mathbb{N}$ *and each* $\varepsilon > 0$, *there are* $n \geqslant 0$ *and a stack* B *of width* $\leqslant 2n$ *such that* $\mu(B) > 1 - \varepsilon$ *and* $\mu(\partial^r B) < \varepsilon\mu(B)$.

By induction, we obtain increasing sequences of CEERs $\mathcal{R}_n$ and positive integers k_n such that $\mathcal{R}_n[\mathcal{T}] \subset \bar{B}_{\mathcal{R}[\mathcal{T}]}(\mathcal{T}, k_n)$ and $\mu(\{\mathcal{T} \in X / \bar{B}_{\mathcal{R}[\mathcal{T}]}(\mathcal{T}, k_{n-1}) \not\subseteq \mathcal{R}_n[\mathcal{T}]\}) < \frac{1}{2^n}$. Define $\mathcal{R}_{n+1}$ as follows: if B is a stack of width $\leqslant 2n \leqslant k_{n+1}$ associated to $r = 2k_n$ and $\varepsilon = 1/2^{n+2}$, then $\mathcal{T}' \in \mathcal{R}_{n+1}[\mathcal{T}]$ if and only if there exists a plaque of B which meets $\mathcal{R}_n[\mathcal{T}]$ and $\mathcal{R}_n[\mathcal{T}']$. Now it is clear that $\mu(\partial\mathcal{R}_n) < 1/2^n$ for each $n \geqslant 0$.

5. Conclusion

Theorem 5.1. *Any equivalence relation* $\mathcal{R}$ *on a Cantor set* X *arising from the continuous hull* $\mathfrak{X}$ *of an aperiodic and repetitive Euclidean tiling is affable.*

Proof. We know that every $\mathcal{R}$-class splits into a finite number of $\mathcal{R}_\infty$-classes. In order to prove the theorem, we will distinguish two cases, depending on whether this number is $\leqslant 2$ or not. In general, the graph of $\mathcal{R}|_{\partial\mathcal{R}_\infty}$ is the union of a countable family of clopen bisections G_i, which are the graphs of partial transformations $\varphi_i : A_i \to B_i$ between disjoints clopen subsets A_i and B_i of $\partial\mathcal{R}_\infty$. Each closed bisection $\bar{G}_i = G_i - \mathcal{R}_\infty$ is the graph of a partial transformation $\bar{\varphi}_i : \bar{A}_i \to \bar{B}_i$ between disjoints closed subsets $\bar{A}_i$ and $\bar{B}_i$ of $\partial\mathcal{R}_\infty$. If all $\mathcal{R}$-classes split at most into two $\mathcal{R}_\infty$-classes, then $\bar{\varphi}_i$ becomes an isomorphism between $\mathcal{R}_\infty|_{\bar{A}_i}$ and $\mathcal{R}_\infty|_{\bar{B}_i}$. For each $i \geqslant 1$, there is a minimal open equivalence subrelation $\mathcal{R}_\infty^{(i)} = \mathcal{R}_\infty \vee \bar{G}_1 \vee \cdots \vee \bar{G}_i = \mathcal{R}_\infty \vee G_1 \vee \cdots \vee G_i \subset \mathcal{R}$ generated by $\mathcal{R}_\infty$ and $\bar{G}_j$, $j \leqslant i$. Applying inductively Theorem 4.18 of [4] (where the condition $\mathcal{R}_\infty^{(i-1)} \cap (\bar{A}_i \times \bar{B}_i) = \emptyset$ may be replaced by $\mathcal{R}_\infty^{(i-1)} \cap \bar{G}_i = \emptyset$, see also [6]), we deduce that $\mathcal{R}_\infty^{(i)}$ and $\mathcal{R} = \bigcup_{i\geqslant 0} \mathcal{R}_\infty^{(i)}$ are affable. In the general case, every point $x \in \bar{A}_i$ has a clopen neighbourhood V_x in A_i such that $\mathcal{R}_\infty[y] = \mathcal{R}_\infty[x]$ iff $\mathcal{R}_\infty[\bar{\varphi}_i(x)] = \mathcal{R}_\infty[\bar{\varphi}_i(y)]$, $\forall y \in V_x$. Indeed, if x and $\bar{\varphi}_i(x)$ belong to two adjacent $\mathcal{P}^{(n)}$-tiles, then y and $\bar{\varphi}_i(y)$ also belong to two adjacent $\mathcal{P}^{(n)}$-tiles in the same flow boxes. In fact, for $p \geqslant n$ big enough, y and $\bar{\varphi}_i(y)$ are in the same $\mathcal{P}^{(p)}$-tiles that x and $\bar{\varphi}_i(x)$ respectively. Therefore $\bar{\varphi}_i : \bar{A}_i \cap V_x \to \bar{B}_i \cap \bar{\varphi}_i(V_x)$ defines an isomorphism between $\mathcal{R}_\infty|_{\bar{A}_i \cap V_x}$ and $\mathcal{R}_\infty|_{\bar{B}_i \cap V_x}$. By compactness of $\bar{A}_i$, each bisection $\bar{G}_i$ decomposes into a finite number of bisections having this property. Finally, by replacing old bisections with new ones, we may conclude as before. $\square$

This theorem is not true for the hyperbolic tilings constructed in [12] because their continuous hulls do not have transverse invariant measures. In an analogous way, to construct a non-uniquely ergodic example arising from a planar tiling, it is enough to decorate the usual tessellation by unit squares with the product of two Oxtoby's sequences [11]. On the other hand, the proof applies to the broader class of *tillable laminations* [2]. So taking the suspension of any free minimal $\mathbb{Z}^m$-action, we have:

Corollary 5.2. *Any free minimal action of* $\mathbb{Z}^m$ *on the Cantor set is affable.*

References

[1] F. Alcalde Cuesta, Á. Lozano Rojo, M. Macho Stadler, Dynamique transverse de la lamination de Ghys–Kenyon, in: Differential Equations and Singularities, 60 years of J.M. Aroca, Astérisque 323 (2009).
[2] J. Bellissard, R. Benedetti, J.M. Gambaudo, Spaces of tilings, finite telescopic approximations and gap-labelling, Comm. Math. Phys. 261 (2006) 1–41.
[3] E. Ghys, Laminations par surfaces de Riemann, Panor. Syntheses 8 (1999) 49–95.
[4] T. Giordano, I. Putnam, C. Skau, Affable equivalence relations and orbit structure of Cantor minimal systems, Ergodic Theory Dynam. Systems 24 (2004) 441–475.
[5] T. Giordano, H. Matui, I. Putnam, C. Skau, Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems, J. Amer. Math. Soc. 21 (2008) 863–892.
[6] T. Giordano, H. Matui, I. Putnam, C. Skau, The absorption theorem for affable equivalence relations, Ergodic Theory Dynam. Systems 28 (2008) 1509–1531.
[7] T. Giordano, H. Matui, I. Putnam, C. Skau, Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems, arXiv:0810.3957.
[8] B. Grünbaum, G.C. Shephard, Tilings and Patterns, W.H. Freeman & Co., New York, 1987.
[9] Á. Lozano Rojo, Dinámica transversa de laminaciones definidas por grafos repetitivos, UPV-EHU Ph.D. thesis, 2008.
[10] H. Matui, Affability of equivalence relations arising from two-dimensional substitution tilings, Ergodic Theory Dynam. Systems 26 (2006) 467–480.
[11] J.C. Oxtoby, Ergodics sets, Bull. Amer. Math. Soc. 58 (1952) 116–136.
[12] S. Petite, On invariant measures of finite affine type tilings, Ergodic Theory Dynam. Systems 26 (2006) 1159–1176.
[13] R.M. Robinson, Undecidability and nonperiodicity of tilings of the plane, Invent. Math. 12 (1971) 177–209.
[14] C. Series, Foliations of polynomial growth are hyperfinite, Israel J. Math. 34 (1979) 245–258.