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Partial Differential Equations

Pathological solutions to elliptic problems in divergence form with continuous coefficients

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Abstract

We construct a function $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ which is a solution to $\operatorname{div}(A\nabla u) = 0$ in the sense of distributions, where A is continuous and $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for $p > 1$. We also give a function $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ such that $u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p < \infty$, u satisfies $\operatorname{div}(A\nabla u) = 0$ with A continuous but $u \notin W^{1,\infty}_{\mathrm{loc}}(B(0,1))$. This answers questions raised by H. Brezis (On a conjecture of J. Serrin, Rend. Lincei Mat. Appl. 19 (2008) 335–338). ***To cite this article: T. Jin et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Solutions pathologiques de problèmes elliptiques sous forme divergence à coefficients continus. Nous construisons une fonction $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$, solution de $\operatorname{div}(A\nabla u) = 0$ au sens des distributions, où A est continu et $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ pour $p > 1$. Nous donnons aussi une fonction $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ telle que $u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ pour tout $p < \infty$, u satisfait $\operatorname{div}(A\nabla u) = 0$ avec A continu mais $u \notin W^{1,\infty}_{\mathrm{loc}}(B(0,1))$. Ceci répond à des questions souleveées par H. Brezis (On a conjecture of J. Serrin, Rend. Lincei Mat. Appl. 19 (2008) 335–338). ***Pour citer cet article : T. Jin et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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On considère l'équation (1), où $\Omega \subset \mathbf{R}^n$ est ouvert. La notion de solution faible est bien définie pour $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ et $A : \Omega \to \mathbf{R}^{n\times n}$ borné, mesurable et elliptique.

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Un résultat fondamental de régularité d'E. De Giorgi [3] dit que si $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, alors u est localement höldérienne. Dans la même direction, N.G. Meyers [8] a prouvé que sous les mêmes hypothèses $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ pour un certain $p > 2$.

J. Serrin [9] a mis en évidence que l'hypothèse $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ était essentielle dans le résultat d'E. De Giorgi en construisant des opérateurs et des solutions dans $W^{1,p}_{\mathrm{loc}}(\Omega)$ avec $p < 2$ qui ne soient pas localement bornées. Suite à cela, il a conjecturé que si les coefficients A étaient höldériens, alors toute solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ était dans $W^{1,2}_{\mathrm{loc}}(\Omega)$.

Cette conjecture a été confirmée pour $u \in W^{1,p}_{\mathrm{loc}}$, avec $1 < p < 2$ par R.A. Hager and J. Ross [4] et pour $u \in W^{1,1}(\Omega)$ by H. Brezis [1,2]. La preuve de H. Brezis s'étend au cas où le module de continuité de A (2) satisfait la condition de Dini (3).

Dans le cas où A est seulement continue, H. Brezis a obtenu le résultat suivant :

Théorème 0.1. *(Voir H. Brezis [1,2].) Supposons que $A \in C(\Omega; \mathbf{R}^{n\times n})$ est elliptique. Si $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ est une solution de* (1)*, alors pour tout $q \in [p, +\infty)$, $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$.*

Ceci conduit H. Brezis a poser deux questions, concernant l'extension de ce résultat aux cas $p = 1$ et $q = +\infty$. Nous donnons une réponse négative à ces deux questions.

Proposition 0.2. *Il existe $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ et $A \in C(B(0,1); \mathbf{R}^{n\times n})$ elliptique tels que u soit une solution de* (1)*, mais $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ pour tout $p > 1$.*

Dans la Proposition 0.2 on peut, en réponse à une question que H. Brezis nous a posée, demander en plus que $Du \in (L \log L)_{\mathrm{loc}}(B(0,1))$. En particulier u est localement dans l'espace de Hardy $\mathcal{H}^1$.

Nous obtenons aussi par la même occasion un résultat de non-unicité, qui répond à une autre question de H. Brezis [1, Open problem 3].

Proposition 0.3. *Il existe $A \in C(\overline{B(0,1)}; \mathbf{R}^{n\times n})$ elliptique tel que le problème* (4) *ait une solution non triviale dans $W^{1,1}_0(B(0,1))$.*

Nous avons enfin

Proposition 0.4. *Il existe $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ et $A \in C(B(0,1); \mathbf{R}^{n\times n})$ elliptique tels que u soit une solution de* (1)*, $u \in W^{1,p}_{\mathrm{loc}}$ pour tout $p \in (1, +\infty)$, mais $u \notin W^{1,\infty}_{\mathrm{loc}}(B(0,1))$.*

On peut de la même manière, en réponse à une question que H. Brezis nous a posée, construire un exemple d'opérateur et de solution tels que $Du \notin \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$.

La preuve repose sur l'identité (5), qui permettent de définir l'opérateur et la solution par (7) et (8) pour la Proposition 0.2 et par (10) et (11) pour la Proposition 0.4. Ces solutions peuvent aussi être obtenues en utilisant des formules asymptotiques de V. Kozlov et V. Maz'ya [6,7].

1. Introduction

Consider the equation

$$-\operatorname{div} A\nabla u = 0 \quad \text{in } \Omega, \tag{1}$$

for $\Omega \subset \mathbf{R}^n$. If $A : \Omega \to \mathbf{R}^{n\times n}$ is bounded, measurable and elliptic, i.e., there exists $\lambda, \Lambda \in \mathbf{R}_*$ such that for every $x \in \Omega$, $A(x)$ is a symmetric matrix, and

$$|\xi|^2 \leqslant \big(A(x)\xi\big) \cdot \xi \leqslant \Lambda|\xi|^2,$$

then one can define a weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ by requiring that for every $\varphi \in C^1_c(\Omega)$,

$$\int_{\Omega} (A\nabla u)\cdot\nabla\varphi = 0.$$

We are interested in the regularity properties of u. A fundamental result of E. De Giorgi [3] states that if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then u is locally Hölder continuous. In particular, u is then locally bounded. In the same direction, N.G. Meyers [8] also proved that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p > 2$.

J. Serrin [9] showed that the assumption $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is essential in E. De Giorgi's result by constructing for every $p \in (1,2)$ a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ that solves such an elliptic equation but which is not locally bounded. In these counterexamples A is not continuous. J. Serrin [9] conjectured that if A was Hölder continuous, then any weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$, and one can then apply E. De Giorgi's theory.

This conjecture was confirmed for $u \in W^{1,p}(\Omega)$ by R.A. Hager and J. Ross [4] and for $u \in W^{1,1}(\Omega)$ by H. Brezis [1,2]. The proof of Brezis extends to the case where the modulus of continuity of A

$$\omega_A(t) = \sup_{\substack{x,y\in\Omega \\ |x-y|\leqslant t}} \bigl|A(x) - A(y)\bigr|, \tag{2}$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_A(s)}{s}\,\mathrm{d}s < \infty. \tag{3}$$

In the case where A is merely continuous, H. Brezis obtained the following result:

Theorem 1.1. *(See H. Brezis [1,2].) Assume that $A \in C(\Omega; \mathbf{R}^{n\times n})$ is elliptic. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ solves* (1)*, then for every $q \in [p, +\infty)$, one has $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$.*

H. Brezis asked two questions about the cases $p = 1$ and $q = \infty$ in the previous theorem. We answer both questions, with a negative answer. First we have

Proposition 1.2. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves* (1)*, but $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$.*

As a byproduct, we obtain, an answer to a further question raised by H. Brezis [1, Open problem 3].

Proposition 1.3. *There exists $A \in C(\overline{B(0,1)}; \mathbf{R}^{n\times n})$ such that the problem*

$$\begin{cases} -\operatorname{div}(A\nabla u) = 0 & \text{in } B(0,1), \\ u = 0 & \text{on } \partial B(0,1) \end{cases} \tag{4}$$

has a nontrivial solution $u \in W^{1,1}_0(B(0,1))$.

H. Brezis asked us the question whether those counterexamples could be improved by taking Du that belongs to $L\log L$ or to the Hardy space $\mathcal{H}^1$. Our construction in Proposition 1.2 answers the question.

Proposition 1.4. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves* (1)*, $Du \in (L\log L)_{\mathrm{loc}}(B(0,1))$ but $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$.*

In particular, in this case, Du belongs locally to the Hardy space $\mathcal{H}^1$ (see [10]).

Concerning the possibility of Lipschitz estimates, we have

Proposition 1.5. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves* (1)*, $Du \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$, $Du \in \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$ but $u \notin W^{1,\infty}_{\mathrm{loc}}(B(0,1))$.*

This shows that $Du \in L^p(B(0,1))$ does not imply $Du \in L^\infty(B(0,1/2))$. H. Brezis asked whether it implies that $Du \in \mathrm{BMO}(B(0,1/2))$. The answer is still negative.

Proposition 1.6. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves* (1)*, $u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p \in (1,\infty)$ but $Du \notin \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$.*

The construction of the counterexamples are made by explicit formulas, inspired by the construction of J. Serrin [9]. They can also be obtained from asymptotic formulas of V. Kozlov and V. Maz'ya [6,7].

2. The pathological solutions

Our counterexamples rely on the following computation:

Lemma 2.1. *Let $v \in C^2((0,R))$ and $\alpha \in C^1((0,R))$. Define $A(x) = (a_{ij}(x))_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}$ by*

$$a_{ij}(x) = \delta_{ij} + \alpha(|x|)\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right).$$

Then for every $x \in B(0,R)\setminus\{0\}$,

$$\operatorname{div}\big(A(x)\nabla\big(x_1 v(|x|)\big)\big) = x_1\left(v''(|x|) + \frac{n+1}{|x|}v'(|x|) - \frac{n-1}{|x|^2}\alpha(|x|)v(|x|)\right). \tag{5}$$

Remark 1. If P is a homogeneous harmonic polynomial of degree k, the formula generalizes to

$$\operatorname{div}\big(A(x)\nabla\big(P(x)v(|x|)\big)\big) = P(x)\left(v''(|x|) + \frac{n+2k-1}{|x|}v'(|x|) - \frac{k(n+k-2)}{|x|^2}\alpha(|x|)v(|x|)\right). \tag{6}$$

Proof of Proposition 1.2. Choose $\beta > 1$, and define for some $r_0 > 1$, for $r \in (0,1)$,

$$v(r) = \frac{1}{r^n(\log\frac{r_0}{r})^\beta}. \tag{7}$$

One takes then

$$\alpha(r) = \frac{r^2v''(r) + (n+1)rv'(r)}{(n-1)v(r)} = \frac{-\beta n}{(n-1)\log\frac{r_0}{r}} + \frac{\beta(\beta+1)}{(n-1)(\log\frac{r_0}{r})^2}. \tag{8}$$

One can take r_0 large enough so that $\alpha \geqslant -\frac{1}{2}$ on $(0,1)$; the coefficient matrix A is then uniformly elliptic. Define now $u(x) = x_1 v(|x|)$. One checks that $u \in W^{1,1}(B(0,1))$ and that u is a weak solution of (1). Indeed, it is a classical solution on $B(0,1)\setminus\{0\}$ by the previous lemma. Taking, $\varphi \in C^1_c(B(0,1))$ and $\rho \in (0,1)$, and integrating by parts we obtain

$$\begin{aligned}\int_{B(0,1)\setminus B(0,\rho)} \nabla\varphi\cdot(A\nabla u) &= -\int_{\partial B(0,\rho)} \varphi\nabla u\cdot\left(A\frac{x}{\rho}\right) = -\int_{\partial B(0,\rho)} \varphi\nabla u\cdot\frac{x}{\rho}\\ &= -\int_{\partial B(0,\rho)} \varphi x_1\left(\frac{v(\rho)}{\rho} + v'(\rho)\right) = -\int_{\partial B(0,\rho)} \big(\varphi(x)-\varphi(0)\big)x_1\left(\frac{v(\rho)}{\rho} + v'(\rho)\right).\end{aligned}$$

Since $\varphi \in C^1_c(B(0,1))$, one has

$$\left|\int_{B(0,1)\setminus B(0,\rho)} \nabla\varphi\cdot(A\nabla v)\right| \leqslant C\rho^n\big(|v(\rho)| + \rho|v'(\rho)|\big),$$

since the right-hand side goes to 0 as $\rho \to 0$, u is a weak solution. □

Remark 2. The examples constructed in the case of merely measurable coefficients by J. Serrin [9] to show that a solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ need not be in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and by N.G. Meyers [8] to show that for every $p > 2$, a solution in $W^{1,2}_{\mathrm{loc}}(\Omega)$ need not be in $W^{1,p}_{\mathrm{loc}}(\Omega)$ can be recovered with the same construction, by taking $v(r) = r^{\alpha}$. The ellipticity condition requires $\alpha < n - 1$ or $\alpha > 1$. This covers all the cases when $n = 2$; a descent argument finishes the construction in higher dimension.

Proof of Proposition 1.4. One checks that the counterexample constructed in the proof of Proposition 1.4 satisfies $Du \in L \log L(B(0, 1))$ when $\beta > 2$. □

Similar examples can be obtained following the results of V. Kozlov and V. Maz'ya [7]. By (4) therein, if $A \in C(B(0, 1); \mathbf{R}^{n\times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $\operatorname{div}(A\nabla u) = 0$ has a solution that is odd with respect to the x_1 variable and that behaves like

$$\frac{x_1}{|x|^n} \exp\Bigl(\int_{B(0,1)\setminus B(0,|x|)} \mathcal{R}(y)\,\mathrm{d}y\Bigr)$$

around 0, where $\mathcal{R}$ is defined following [7, (3)][1]

$$\mathcal{R}(x) = \frac{(e_1 \cdot (A(x) - A(0))e_1)(x \cdot A(0)^{-1}x) - n(e_1 \cdot (A(x) - A(0))A(0)^{-1}x)(e_1 \cdot x)}{|\partial B(0, 1)||\det A(0)|^{\frac{1}{2}}(x \cdot A(0)^{-1}x)^{\frac{n}{2}+1}}. \tag{9}$$

Taking A as in Lemma 2.1 with $\lim_{r\to 0} \alpha(r) = 0$, one has $\mathcal{R}(x) = \alpha(|x|)(|x|^2 - x_1^2)/(|\partial B(0, 1)||x|^{n+2})$. Therefore, there is a solution that behaves like

$$\frac{x_1}{|x|^n} \exp\Biggl(\frac{n-1}{n}\int_{|x|}^{1} \alpha(r)\,\frac{\mathrm{d}r}{r}\Biggr).$$

In particular, if one takes $\alpha(r) = -\beta n/((n - 1)\log \frac{r_0}{r})$, one obtains a solution that behaves like $\frac{x_1}{|x|^n}(\log \frac{r_0}{r})^{-\beta}$. One could also take $a_{ij}(x) = \delta_{ij} + \kappa(|x|)(\delta_{ij} - n\delta_{i1}\delta_{j1}\frac{x_1^2}{|x|^2})$ and continue the computations with now $\mathcal{R}(x) = \kappa(|x|)(|x|^2 - nx_1^2)^2/(|\partial B(0, 1)||x|^{n+2})$.

Proof of Proposition 1.3. This follows H. Brezis [1, §A.5]. Let u be given by the proof of Proposition 1.2. Notice that u is smooth on $\partial B(0, 1)$. Since A is bounded and elliptic, the problem

$$\begin{cases} -\operatorname{div}(A\nabla w) = 0 & \text{in } B(0, 1),\\ w = u & \text{on } \partial B(0, 1)\end{cases}$$

has a unique solution in $w \in W^{1,2}(B(0, 1))$. Since $u \notin W^{1,2}(B(0, 1))$, $u \neq w$. Hence, $u - w \in W^{1,1}(B(0, 1))$ is a nontrivial solution of (4). □

Proof of Proposition 1.5. Take for $r \in (0, 1)$,

$$v(r) = \log \frac{r_0}{r} \tag{10}$$

and

$$\alpha(r) = \frac{1 - (n + 1)}{(n - 1)\log \frac{r_0}{r}} = \frac{-n}{(n - 1)\log \frac{r_0}{r}}, \tag{11}$$

[1] The reader should correct the misprint in [7, (3)] and read $|S^{n-1}_{+}|$ instead of $|S^{n-1}|$.

where r_0 is chosen so that $\alpha(r) > -\frac{1}{2}$ on $(0,1)$. Defining $u(x) = x_1 v(|x|)$, one checks that $Du \in W^{1,p}_{\mathrm{loc}}(B(0,1))$, $Du \in \mathrm{BMO}(B(0,1))$, $u \notin W^{1,\infty}(B(0,1))$ and that u solves (1) in the sense of distributions. □

As for the previous singular pathological solutions, similar examples can be obtained from results of V. Kozlov and V. Maz'ya for solutions [6]. By (4) therein if $A \in C(B(0,1); \mathbf{R}^{n\times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $\mathrm{div}(A\nabla u) = 0$ has a solution in $W^{1,2}(B(0,1))$ that is odd with respect to the x_1 variable and that behaves like

$$x_1 \exp\left(-\int_{B(0,1)\setminus B(0,|x|)} \mathcal{R}(y)\,\mathrm{d}y\right)$$

around 0, where $\mathcal{R}$ is given by (9). Taking A as in Lemma 2.1 with $\alpha(r) = \frac{-n}{n-1}(\log\frac{r_0}{r})^{-1}$ one recovers the counterexample presented above.

Proof of Proposition 1.6. Define for $r \in (0,1)$,

$$v(r) = \left(\log\frac{r_0}{r}\right)^2$$

and

$$\alpha(r) = \frac{-2n}{(n-1)\log\frac{r_0}{r}} + \frac{2}{(n-1)(\log\frac{r_0}{r})^2}.$$

Defining $u(x) = x_1 v(|x|)$, one checks that $u \in W^{1,p}(B(0,1))$ for every $p > 1$ and that u solves (1) in the sense of distributions. One checks that for every $c > 0$, $\exp(c|Du|) \notin L^1(B(0,\frac{1}{2}))$; hence by the John–Nirenberg embedding theorem [5] (see also e.g. [11, Chapter 4, §1.3]), $Du \notin \mathrm{BMO}(B(0,\frac{1}{2}))$. □

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