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# A numerical study of the null boundary controllability of a convection diffusion equation

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**Abstract**

In this paper we study numerically the cost of the null controllability of a linear control parabolic 1-D equation as the diffusion coefficient tends to 0. For this linear control parabolic 1-D equation, we know from a prior work by J.-M. Coron and S. Guerrero (2005), that, when the diffusion coefficient tends to 0, for a small controllability time, the norm of the optimal control tends to infinity and that, if the controllability time is large enough, this norm tends to 0. For controllability times which are not covered by this work, we estimate numerically the norm of the optimal control as the diffusion coefficient tends to 0. ***To cite this article: A. Salem, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


**Résumé**

**Étude numérique de la contrôlabilité frontière d'une équation de convection diffusion.** Dans cette Note nous étudions de manière numérique le coût de la contrôlabilité à zéro d'une équation de convection diffusion unidimensionnelle quand le coefficient de diffusion tend vers 0. Nous savons, d'après un travail antérieur de J.-M. Coron et S. Guerrero (2005), que pour un temps de contrôlabilité trop petit la norme du contrôle optimal tend vers l'infini quand le terme de diffusion tend vers zéro et que, par contre, cette norme tend vers 0 si le temps de contrôlabilité est assez grand. Pour des valeurs du temps de contrôle qui ne sont pas couvertes par ce travail, nous étudions numériquement la norme du contrôle optimal quand le coefficient de diffusion tend vers 0. ***Pour citer cet article : A. Salem, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


## Version française abrégée

Soit le problème de contrôle linéaire (2) où $u \in L^2(0,1)$ est l'état du système et $v \in \mathbb{R}$ est le contrôle. Il est bien connu que le système (2) est contrôlable à zéro au temps $t = T$. Cela signifie que pour tout $u^0 \in L^2(0,1)$ et pour tout $(\epsilon, T) \in (0, +\infty)^2 \times \mathbb{R}$, il existe $v \in L^2(0,T)$ tel que la solution de (2) vérifie $u(., T) = 0$. Ce résultat est dû à Fattorini et Russell [6]. Voir aussi Fursikov–Imanuvilov [7] et Lebeau–Robbiano [9] pour des dimensions d'espace plus grandes que 1. Pour $u^0 \in L^2(0,L)$, nous définissons $U(\epsilon, T, u^0)$ l'ensemble des contrôles $v \in L^2(0,T)$ tels que

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la solution de (2) vérifie $u(., T) = 0$. Nous définissons aussi la quantité qui mesure le coût de la contrôlabilité à zéro du système (2) :

$$K(\epsilon, T) = \sup_{\|u^0\|_{L^2(0,1)} \leqslant 1} \left\{ \min\left\{ \|v\|_{L^2(0,T)} \colon v \in U\left(\epsilon, T, u^0\right)\right\}\right\}. \tag{1}$$

**Remarque 1.** Il est facile de voir que $U(\epsilon, T, u^0)$ est un sous-ensemble fermé affine de $L^2(0, T)$. Ainsi le minimum dans (1) est atteint.

Dans [2] J.-M. Coron et S. Guerrero ont étudié le comportement de $K(\epsilon, T)$ quand $\epsilon \to 0^+$. Ils ont montré que, si le temps de contrôlabilité $T \geqslant 4.3$, $K(\epsilon, T)$ tend vers zéro quand $\epsilon$ tend vers 0 et que, par contre, si $T$ est trop petit, $K(\epsilon, T)$ tend vers $+\infty$ quand $\epsilon$ tend vers 0 (avec une estimation de ces convergences).

Le but de cette Note est d'étudier de manière numérique $K(\epsilon, T)$ quand $\epsilon$ tend vers 0 pour des valeurs de $T > 1$ trop petites pour être couvertes par [2].

Nous appliquons la méthode HUM introduite par J.-L. Lions dans [10] comme algorithme de calcul pour chercher contrôle optimal. L'équation (2) est discrétisée à l'aide de la méthode aux differences finies. Nous utilisons un schéma comme celui introduit par S. Ervedoza, C. Zheng et E. Zuazua dans [5].

## 1. Introduction

Let $(\epsilon, T) \in (0, +\infty)^2$. We consider the following parabolic linear control system:

$$\begin{cases} u_t - \epsilon u_{xx} + u_x = 0 & \text{in } (0,1) \times (0,T), \\ u(0,t) = v(t),\ u(1,t) = 0 & \text{on } (0,T), \\ u(x,0) = u^0(x) & \text{in } (0,1), \end{cases} \tag{2}$$

where the state is $u(., t) \in L^2(0, 1)$ and the control is $v(t) \in \mathbb{R}$.

It is known that system (2) is controllable to the null final state at time $t = T$. This means that, for every $u^0 \in L^2(0, 1)$ and for every $(\epsilon, T) \in (0, +\infty)^2$, there exists $v(t) \in L^2(0, T)$ such that the (weak) solution of (2) satisfies $u(T, .) = 0$. This result is due to Fattorini and Russell [6]. See also Fursikov–Imanuvilov [7] and Lebeau–Robbiano [9] for parabolic control systems in dimension larger than 1. We are interested to study the cost dependance of the null controllability of system (2) with respect to the two parameters $\epsilon$ and $T$.

**Definition 1.1.** For $u^0 \in L^2(0, 1)$, we denote by $U(\epsilon, T, u^0)$ the set of controls $v \in L^2(0, T)$ such that the corresponding solution of (2) satisfies $u(., T) = 0$. Next, we define the quantity which measures the cost of the null controllability for system (2):

$$K(\epsilon, T) := \sup_{\|u^0\|_{L^2(0,1)} \leqslant 1} \left\{ \min\left\{ \|u\|_{L^2(0,T)} \colon u \in U\left(\epsilon, T, u^0\right)\right\}\right\}. \tag{3}$$

**Remark 1.** For $\epsilon = 0$, the control system (2) is controllable in time $T$ if and only if $T \geqslant 1$.

In [2] J.-M. Coron and S. Guerrero have studied the behavior of $K(\epsilon, T)$ as $\epsilon \to 0$. In particular, they have proved the following theorem:

**Theorem 1.** *(See [2].)*

- *If* $T < 1$ *one has* $\lim_{\epsilon \to 0^+} K(\epsilon, T) = +\infty$.
- *If* $T \geqslant 4.3$ *one has* $\lim_{\epsilon \to 0^+} K(\epsilon, T) = 0$.

In Theorem 1 the constant 4.3 is not optimal. The goal of this Note is to study numerically $K(\epsilon, T)$ as $\epsilon \to 0$ for $T > 1$ which are too small to use [2].

The problem (2) is discretized using the difference finite method. Our numerical scheme is similar to the one introduced by S. Ervedoza, C. Zheng and E. Zuazua, [5]. The HUM method is implemented to find the optimal control.

The Note is organized as follows: In Section 2 we describe the numerical algorithm based on HUM for boundary controllability of the parabolic linear control system (2). In Section 3 we describe the numerical scheme that has been implemented. In Section 4 we present several numerical simulations and we investigate the relations between the control time $T$ and the control cost as $\epsilon \to 0^+$.

## 2. The numerical algorithm based on HUM

### 2.1. Problem formulation

We will study from a numerical viewpoint the following null boundary controllability problem: Given $T > 0$, $u^0(x) \in L^2(0,1)$ and $\epsilon > 0$ we search for a control function $v \in L^2(0,T)$ such that the solution $u$ of the boundary initial-value problem (2) satisfies $u(x,T) = 0$. In the present setting, this result is equivalent to an observability inequality for the adjoint equation:

$$\begin{cases} \varphi_t + \varphi_{xx} + \epsilon\varphi_x = 0 & \text{in } (0,1)\times(0,T), \\ \varphi(0,t) = \varphi(1,t) = 0 & \text{on } (0,T), \\ \varphi(x,T) = \varphi^0(x) & \text{in } (0,1). \end{cases} \tag{4}$$

More precisely, it is equivalent to the existence of a positive constant $C > 0$ such that:

$$\|\varphi(0)\|^2_{L^2(0,1)} \leqslant C \int_0^T |\varphi_x(0,t)|^2. \tag{5}$$

Note that the infimum of $C > 0$ such that (5) holds is $\epsilon^2 K(\epsilon, T)$ (see [1, Remark 2.98]).

J.-L. Lions presented in [10] a constructive method allowing the calculation of control $v$, namely the Hilbert Uniqueness Method (HUM). It gives the control with minimal $L^2$-norm. In order to apply the HUM algorithm, we minimize the following functional $J$:

$$J(\varphi^0) = \frac{\epsilon}{2}\int_0^T \varphi_x^2(0,t)\,\mathrm{d}t - \int_0^1 \varphi(x,0)u^0(x)\,\mathrm{d}x \tag{6}$$

on the Hilbert space

$$H = \{\varphi^0 \in L^2(0,1),\ \text{where } \varphi \text{ is solution of (4) and } \varphi_x(0,t) \in L^2(0,T)\}. \tag{7}$$

The functional $J$ is continuous and, from the observability inequality (5), is strictly convex and coercive in $H$. Thus it has a minimum. The HUM goes as follows:

- Find $\varphi_0$ the minimum of $J$ on the Hilbert space $H$, then solve problem (4).
- The control is then $v(t) = -\varphi_x(0,t)$.

### 2.2. The HUM based algorithm

The previous paragraph proved that the crucial point in characterizing the control $v$ is to minimize the functional (6). By deriving $J$ along $\psi$, we get

$$J'(\varphi^0)\psi^0 = \int_0^T \epsilon\varphi_x(0,t)\psi_x(0,t)\,\mathrm{d}t - \int_0^1 \psi(x,0)u^0(x)\,\mathrm{d}x, \tag{8}$$

where $\psi$ is the solution of the following adjoint problem:

$$\begin{cases} \psi_t + \psi_{xx} + \epsilon\psi_x = 0 & \text{in } (0,1)\times(0,T), \\ \psi(0,t) = \psi(1,t) = 0 & \text{on } (0,T), \\ \psi(x,T) = \psi^0(x) & \text{in } (0,1). \end{cases} \tag{9}$$

We deduce a necessary condition so that (6) has a minimum:

$$\int_0^T \epsilon\varphi_x(0,t)\psi_x(0,t)\,\mathrm{d}t - \int_0^1 \psi(x,0)u^0(x)\,\mathrm{d}x = 0. \tag{10}$$

We then write problem (10) on the following variational form:

$$\text{Find } \varphi \in H \text{ such that } a(\varphi,\psi) = L(\psi),\ \forall \psi \in H, \tag{11}$$

where

$$a(\varphi,\psi) = \int_0^T \epsilon\varphi_x(0,t)\psi_x(0,t)\,\mathrm{d}t$$

and

$$L(\psi) = \int_0^1 \psi(x,0)u^0(x)\,\mathrm{d}x.$$

It is clear that $a$ is a bilinear continuous symmetric functional. By the inequality observability (5), $a$ is coercive. We deduce that problem (11) can be solved using the conjugate gradient algorithm.

### 2.3. *Problem discretization*

In order to make a discretization of our problem, we are inspired by the method proposed by S. Ervedoza, C. Zheng and E. Zuazua in [5]. For each $N \in \mathbb{N}$, we consider a partition of $(0, L)$, $P = \{x_0 = 0, \dots, x_j = jh, \dots, x_{N+1} = 1\}$ where $h = 1/(N+1)$ is the space step. We denote $u_j(t)$ the approximation of the solution to point $x_j$. Let $\delta t$ be the time step and $u_j^n = u(n\delta t, jh)$. Then, using implicit scheme, the problem of semi-discretization is the following system of $N$ ordinary differential equations:

$$\begin{cases} \frac{1}{(\delta t)}(u_j^{n+1} - u_j^n) + \frac{1}{2h}(u_{j+1}^{n+1} - u_j^{n+1}) + \frac{1}{2h}(u_{j+1}^n - u_j^n) \\ \quad - \frac{\epsilon}{2h^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) - \frac{\epsilon}{2h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0, & j = 1,\dots,N,\ n = 1,\dots I, \\ u_0^{n+1} = v(n\delta t),\ u_{N+1}^{n+1} = 0, & n = 1,\dots,I, \\ u_j(0) = u_j^0, & j = 1,\dots,N. \end{cases} \tag{12}$$

We define the matrices $A$ and $B$ by

$$A(i,i) = -2, \quad A(i,i-1) = 1 \quad \text{and} \quad A(i,i+1) = 1,$$
$$B(i,i) = -1 \quad \text{and} \quad B(i,i+1) = 1.$$

Also, let $Y^n$ be a vector from $\mathbb{R}^N$ of components $u_j^n$ and let $G^n$ be a vector from $\mathbb{R}^N$ defined by $G^n = (0,\dots,0,g(n\delta t))$. The previous scheme becomes:

$$\frac{1}{(\delta t)}\big(Y^{n+1} - Y^n\big) = \frac{\epsilon}{2h^2}A\big(Y^{n+1} + Y^n\big) - \frac{1}{2h}B\big(Y^{n+1} + Y^n\big) - \frac{\epsilon}{2h^2}\big(G^{n+1} + G^n\big).$$

It follows from [4] that this implicit scheme is stable. The discretization of adjoint system is:

$$\begin{cases} \frac{1}{(\delta t)}(\varphi_j^{n+1} - \varphi_j^n) + \frac{1}{2h}(\varphi_{j+1}^{n+1} - \varphi_j^{n+1}) + \frac{1}{2h}(\varphi_{j+1}^n - \varphi_j^n) \\ \quad + \frac{\epsilon}{2h^2}(\varphi_{j+1}^{n+1} - 2\varphi_j^{n+1} + \varphi_{j-1}^{n+1}) + \frac{\epsilon}{2h^2}(\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n) = 0, & j = 1,\dots,N,\ n = 1,\dots,I, \\ \varphi_0^{n+1} = \varphi_{N+1}^{n+1} = 0, & n = 1,\dots,I, \\ \varphi_j(T) = \varphi_j^0, & j = 1,\dots,N. \end{cases} \tag{13}$$

The semi-discretization of the functional (6) is:

$$J(\varphi^0) = \frac{\epsilon}{2}\int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 \mathrm{d}t - \int_0^1 \varphi(x,0)u^0(x)\,\mathrm{d}x, \tag{14}$$

where $\varphi = (\varphi_1, \ldots, \varphi_N)$ is the solution of problem (4) having the initial condition $\varphi^0 = (\varphi_1^0, \ldots, \varphi_N^0)$ and $u^0 = (u_1^0, \ldots, u_N^0)$ is the initial condition of system (12). According to the work of J.-M. Coron and S. Guerrero [2] combined with the article of S. Labbé and E. Trélat [8] and the paper of S. Ervedoza, C. Zheng and E. Zuazua [5] we can conclude the uniform controllability with respect of $\delta t$, $h$ and $\epsilon$ of the control system (12) for $T \geqslant 4.3$.

**Remark 2.** Our numeral scheme is implicit. If one replaces, in (12), $(1/(2h)(u_{j+1}^{n+1} - u_j^{n+1}) + (1/(2h))(u_{j+1}^n - u_j^n)$ by $(1/h)(u_{j+1}^n - u_j^n)$, one knows that, at least for $\epsilon = 0$, the numerical scheme generates high frequency spurious waves which propagate at zero speed making the controllability non-uniform with respect to $\epsilon$. See [3].

## 3. Numerical study

These numerical experiments have been made with the following conditions:

- The initial condition is $(\sin(\pi x))^2$;
- The gradient stopping condition is $10^{-12}$;
- The space step discretization is $h = 10^{-2}$;
- The time space discretization is $\delta t = 10^{-2}$.

We see in Figs. 1 to 4 that, for every $T \geqslant 1.1$, the $L^2(0,T)$ control norm tends to zero as $\epsilon$ tends to 0. We note that the necessary iteration number $N$ is independent of the discretization step $h$. We can note that the iterations number becomes constant starting from $T = T' \simeq 1.6$. It reaches a value close to $N = 20$ which allows us to state the following conjecture:

**Conjecture 2.** *There exist $T' \in (1, 1.6)$ such that*

- *$\forall T > T'$, the control system (12) is uniformly controllable with respect to $\delta t$, $\epsilon$ and $h$.*
- *$\forall T \in (1, T')$, the control system (12) is uniformly controllable with respect to $\delta t$, $h$ but not with respect to $\epsilon$.*

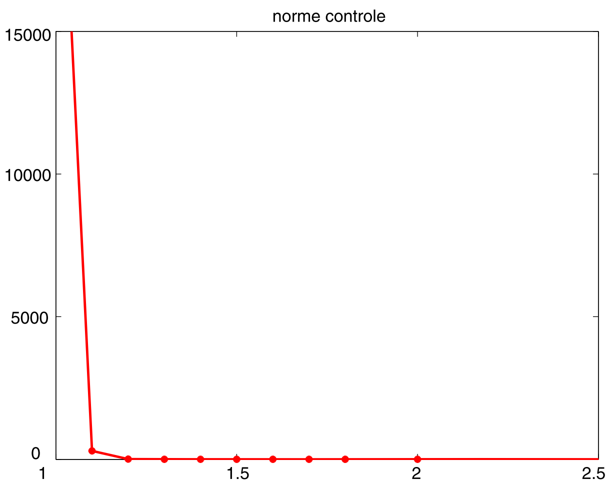


Fig. 1. $\|v(t)\|_{L^2}$ with $\epsilon = 0.01$.

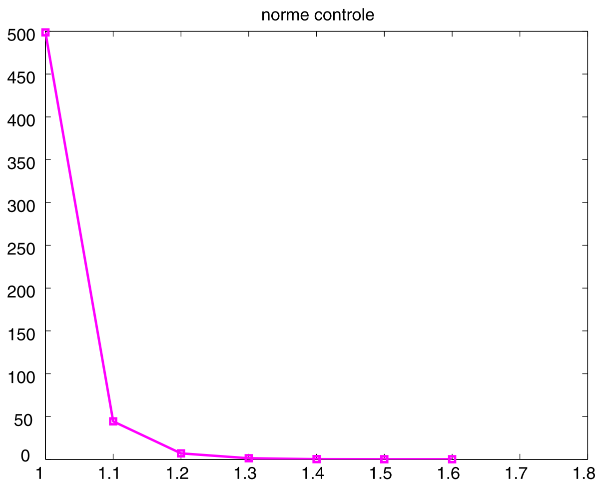


Fig. 2. $\|v(t)\|_{L^2}$ with $\epsilon = 0.025$.

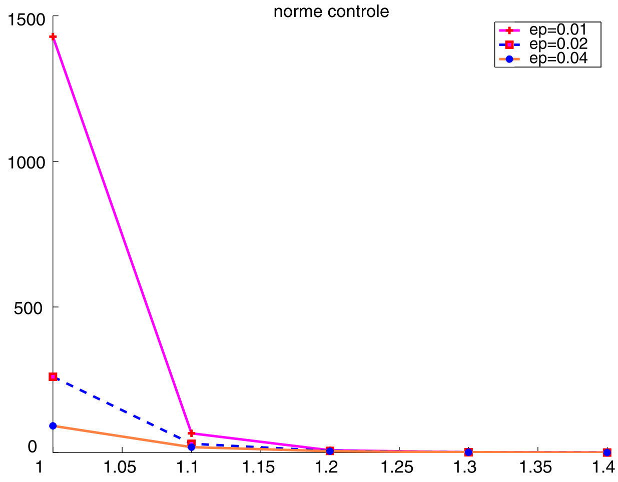


Fig. 3. $\|v(t)\|_{L^2}$ for different values of $\epsilon$.

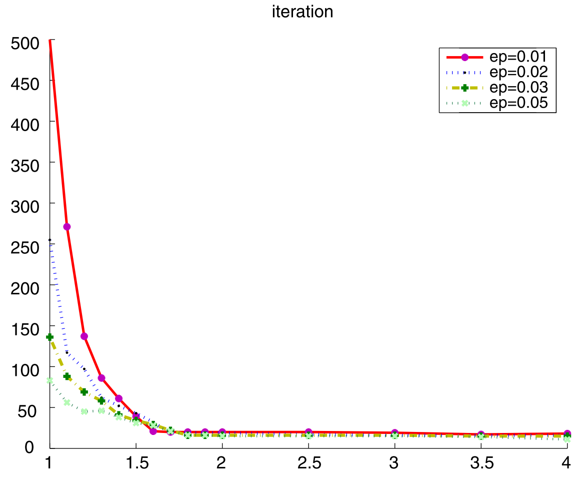


Fig. 4. Number of iterations.

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