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Partial Differential Equations

On the boundary controllability of non-scalar parabolic systems

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Abstract

This Note is concerned with the boundary controllability of non-scalar linear parabolic systems. More precisely, two coupled one-dimensional linear parabolic equations are considered. We show that, with boundary controls, the situation is much more complex than for similar distributed control systems. In our main result, we provide necessary and sufficient conditions for null controllability. ***To cite this article: E. Fernández-Cara et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur la contrôlabilité frontière des systèmes paraboliques non scalaires. Cette Note concerne la contrôlabilité frontière des systèmes paraboliques linéaires non scalaires. Plus précisement, on considère un système de deux équations paraboliques linéaires de dimension 1 en espace. Nous montrons qu'il est beaucoup plus compliqué de contrôler sur une partie du bord que de le faire avec des contrôles distribués. Dans notre résultat principal, on donne des conditions nécessaires et suffisantes pour la contrôlabilité exacte à zéro. ***Pour citer cet article : E. Fernández-Cara et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction and main result

Let us fix $T > 0$ and let us consider the linear system

$$\begin{cases} y_t - y_{xx} = Ay & \text{in } Q = (0,1) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(1,\cdot) = 0 & \text{in } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,1), \end{cases} \tag{1}$$

where $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are given and $y_0 \in H^{-1}(0,1)^2$. Here, $v \in L^2(0,T)$ is a control function (to be determined) and $y = (y_1, y_2)^*$ is the state variable. Observe that, for every $v \in L^2(0,T)$ and $y_0 \in H^{-1}(0,1)^2$, (1) admits a unique weak solution (defined by transposition) that satisfies $y \in L^2(Q)^2 \cap C^0([0,T]; H^{-1}(0,1)^2)$.

It will be said that (1) is *approximately controllable* in $H^{-1}(0,1)^2$ at time T if, for any $y_0, y_d \in H^{-1}(0,1)^2$ and any $\varepsilon > 0$, there exists a control function $v \in L^2(0,T)$ such that the associated solution satisfies $\|y(\cdot,T) - y_d\|_{H^{-1}(0,1)} \leqslant \varepsilon$.

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It will be said that (1) is *null controllable* at time T if, for each $y_0 \in H^{-1}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that

$$y(\cdot,T) = 0 \quad \text{in } H^{-1}(0,1)^2. \tag{2}$$

Since (1) is linear, it is null controllable if and only if it is *exactly controllable to the trajectories* at time T. That is to say, if and only if for any solution y^* to (1) corresponding to $v \equiv 0$ and $y_0^* \in H^{-1}(0,1)^2$ and any $y_0 \in H^{-1}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that the associated solution satisfies

$$y(\cdot,T) = y^*(\cdot,T) \quad \text{in } H^{-1}(0,1)^2.$$

The controllability properties of similar scalar problems are nowadays well known; see for instance [7,14,6,13,10] and [8]. More precisely, let $\Omega \subset \mathbb{R}^N$ be a nonempty regular bounded open set with $N \geqslant 1$, let $\omega \subset \Omega$ be a nonempty open subset, and let $\gamma \subset \partial\Omega$ be a nonempty relative open set. Let us consider the following scalar problems:

$$\begin{cases} y_t - \Delta y = v1_\omega & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega \end{cases} \qquad \text{and} \qquad \begin{cases} y_t - \Delta y = 0 & \text{in } \Omega \times (0,T), \\ y = v1_\gamma & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

where 1_ω and 1_γ are, respectively, the characteristic functions of ω and γ and $y_0 \in L^2(\Omega)$ is given. Then, for every Ω, ω, γ and T, both systems are approximately controllable in $L^2(\Omega)$ and null controllable at time T. In fact, the boundary controllability results for the system in the right can be easily obtained from the corresponding distributed controllability results for system in the left and vice versa. But, as we will show, the situation is quite different for similar non-scalar systems.

There are not many works devoted to the controllability of parabolic systems of PDEs. To our knowledge, all them deal with distributed controls, exerted on a small open set ω; see for instance [15,5,1,4,11,12,2] and [3]. In particular, in [2] and [3], the authors consider the system

$$\begin{cases} y_t - \Delta y = Ay + Bv1_\omega & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{3}$$

where $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$ (with $n, m \geqslant 1$) and $y_0 \in L^2(\Omega)^n$. They prove that (3) is null controllable if and only the following *Kalman's rank condition* is satisfied:

$$\operatorname{rank}[A \mid B] = n. \tag{4}$$

Here, we have used the notation $[A \mid B] := [B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B]$.

The main goal of this Note is to characterize the boundary controllability properties of (1) (a system of 2 equations) when we apply just one control on a part of the boundary. Our main result is the following:

Theorem 1.1. *Let $A \in \mathcal{L}(\mathbb{R}^2;\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A. Then* (1) *is null controllable at any time $T > 0$ if and only if one has* (4) (*with $n = 2$*) *and*

$$\pi^{-2}(\mu_1 - \mu_2) \neq j^2 - k^2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j. \tag{5}$$

Thus, we observe that the Kalman's rank condition (4) is necessary, but not sufficient, for the boundary controllability of (1) (unlike the distributed case). This is a crucial discrepancy between boundary and distributed controllability for coupled parabolic systems and shows that, for a given system, these two properties can be independent.

In view of Theorem 1.1, we find two different situations: when the matrix A in (1) has a double real eigenvalue or a couple of conjugate complex eigenvalues, (4) is a necessary and sufficient condition for the null controllability at any time (as in the distributed case). Otherwise, if A has two different real eigenvalues, an additional condition is needed for null controllability, independently of the vector B we are considering. Furthermore, the conditions (4) and (5) are also equivalent to the approximate controllability of (1) at any time T.

The proof of Theorem 1.1 is given in [9]. It relies on a method by Fattorini and Russell that was used in [7] to prove for the first time the boundary null controllability of the one-dimensional heat equation. The key point in the proof is the following lemma:

Lemma 1.2. *Suppose that $\{\Lambda_n\}_{n\geqslant 1}$ is a sequence of complex numbers such that*

$$\Re(\Lambda_n) \geqslant \delta|\Lambda_n| \quad \forall n \geqslant 1, \quad |\Lambda_n - \Lambda_k| \geqslant |n-k|\rho \quad \forall n,k \geqslant 1 \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{|\Lambda_n|} < +\infty, \tag{6}$$

for some $\delta, \rho > 0$. Then there exists a sequence $\{q_n, \tilde{q}_n\}$ that is biorthogonal to the family $\{e^{-\Lambda_n t}, te^{-\Lambda_n t}\}$ and such that, for every $\varepsilon > 0$, one has:

$$\big\|(q_n, \tilde{q}_n)\big\|_{L^2(0,+\infty)} \leqslant K(\varepsilon) e^{\varepsilon \Re(\Lambda_n)} \quad \forall n \geqslant 1. \tag{7}$$

Of course, that $\{q_n, \tilde{q}_n\}$ is biorthogonal to $\{e^{-\Lambda_n t}, te^{-\Lambda_n t}\}$ means that $(q_n, e^{-\Lambda_k t}) = \delta_{kn}$, $(q_n, te^{-\Lambda_k t}) = 0$, $(\tilde{q}_n, e^{-\Lambda_k t}) = 0$ and $(\tilde{q}_n, te^{-\Lambda_k t}) = \delta_{kn}$ for all $n, k \geqslant 1$. The proof of this lemma is given in [9] (a similar but simpler result was given in [7]).

The proof of Theorem 1.1 relies on an appropriate observability inequality for the solutions to the adjoint system to (1). The main point is to prove inequalities of the kind

$$\int_0^T \Big|\sum_{j\geqslant 1} C_j e^{-\Lambda_j t}\Big|^2 \,\mathrm{d}t \geqslant C_T \sum_{j\geqslant 1} \frac{C_j^2}{|\Lambda_j|} e^{-\Lambda_j T}$$

and

$$\int_0^T \Big|\sum_{j} (A_j + tB_j) e^{-\Lambda_j t}\Big|^2 \,\mathrm{d}t \geqslant C_T \sum_{j\geqslant 1} \frac{(A_j + TB_j)^2}{|\Lambda_j|} e^{-\Lambda_j T}$$

and use them for some particular values of Λ_n related to the eigenvalues of the Dirichlet Laplacian and the eigenvalues μ_1 and μ_2 of A.

2. Some additional comments

It would be very interesting to generalize Theorem 1.1 to the case of a $n \times n$ coupled system (with $n \geqslant 3$), controlled by m boundary control forces ($m \geqslant 1$). Thus, let us assume that, in (1), $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ and $y_0 \in H^{-1}(0,1)^n$. It can be proved that the Kalman's rank condition (4) is necessary for the approximate and exact controllability to the trajectories of system (1) at time T (for a proof, see for instance [2] and [3]). But, as in the case $n = 2$, there are some necessary conditions (independent of B) that arise in the study of the controllability properties of this $n \times n$ system.

Indeed, let us assume that $n \geqslant 3$ and for instance $m = 1$, i.e. $B \in \mathbb{R}^n$. Let us also suppose that there exist two integers $j_0, k_0 \geqslant 1$ with $j_0 \neq k_0$ and two eigenvalues $\mu, \widetilde{\mu} \in \mathbb{C}$ of A such that

$$\pi^{-2}(\mu - \widetilde{\mu}) = j_0^2 - k_0^2. \tag{8}$$

Then, it can be proved that (1) is neither null nor approximately controllable in $H^{-1}(\Omega)^n$ at time T. In other words, (8) is a necessary condition for the controllability of system (1) when $n \geqslant 3$ and $m = 1$.

A more complete analysis of this general situation will be the goal of a forthcoming paper.

Finally, let us consider a situation where distributed and boundary controls do not play the same role. This will give an idea of the, in some sense, unnatural difficulties that arise when we try to control a non-scalar system from the boundary. Thus, let us consider the following cascade system, where $\nu > 0$ and $Q = (0,T) \times (0,1)$:

$$\begin{cases} y_t - \nu y_{xx} = 0 & \text{in } Q, \\ y(t,0) = v, \quad y(t,1) = 0 & \text{in } (0,T), \\ y(.,0) = y^0 & \text{in } (0,1), \end{cases} \qquad \begin{cases} q_t - q_{xx} = y & \text{in } Q, \\ q(t,0) = q(t,1) = 0 & \text{in } (0,T), \\ q(.,0) = q^0 & \text{in } (0,1). \end{cases} \tag{9}$$

We address the following approximate controllability question: Let $\varepsilon > 0$, $(y^0, q^0) \in L^2(0,1) \times L^2(0,1)$ and $(y^1, q^1) \in L^2(0,1) \times L^2(0,1)$ be given; then, does there exist $v \in L^2(0,T)$ such that the corresponding solution to (9) satisfies

$$\big\|y(T) - y^1\big\|_{L^2} + \big\|q(T) - q^1\big\|_{L^2} \leqslant \varepsilon?$$

We have the following result (see [9] for the proof):

Theorem 2.1. *Suppose that* $\nu \neq 1$. *Then* (9) *is approximately controllable at time* $T > 0$ *if and only if* $\sqrt{\nu} \notin \mathbb{Q}$.

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