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Analytic Geometry

On an analog of Pinkham's theorem for non-Tjurina components of rational singularities

Selma Altınok

Adnan Menderes Üniversitesi, Matematik Bölümü, 09100 Aydın, Turkey

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Abstract

There is a correspondence between the set of functions in the maximal ideal of the local ring of a rational surface singularity ξ and the set $\mathcal{E}^+(E)$ consisting of certain effective divisors supported on the exceptional fiber E of a resolution of the singularity. Given an element $Y \in \mathcal{E}^+(E)$ and a non-Tjurina component N of Y, we verify a formula for the least element of the set of divisors $X \in \mathcal{E}^+(E)$ greater than or equal to $Y + N$ stated but not proved in Tosun (1999). ***To cite this article: S. Altınok, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur un analogue du théorème de Pinkham pour les composantes des singularités rationnelles qui ne sont pas Tjurina. Il existe une correspondance entre l'ensemble des fonctions de l'idéal maximal de l'anneau local en une singularité rationnelle ξ d'une surface et un ensemble $\mathcal{E}^+(E)$ de diviseurs effectifs portés par la fibre exceptionnelle E d'une résolution de cette singularité. Étant donné un élément $Y \in \mathcal{E}^+(E)$ et une composante N de Y qui n'est pas Tjurina, nous établissons une formule donnant le plus petit élément de l'ensemble des diviseurs $X \in \mathcal{E}^+(E)$ supérieur ou égal à $Y + N$, indiquée mais non démontrée dans Tosun (1999). ***Pour citer cet article : S. Altınok, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let (X, ξ) be a germ of a normal analytic surface embedded in $\mathbb{C}^n$ having a rational singularity at ξ. Such a singularity is called rational if $H^1(\widetilde{X}, \mathcal{O}_X) = 0$, where $\pi : \widetilde{X} \to X$ is any resolution of X at ξ. Since there is a correspondence between analytic functions on X and certain positive divisors supported on the exceptional fiber of a resolution of the singularity (see [2] or [5]), the study of such divisors is useful in order to obtain information about the structure of analytic functions. These divisors can be used to calculate certain topological invariants such as Seiberg–Witten invariants of the plumbed manifold corresponding to a singularity (see [6]), also to read the open book structure of this manifold (see [3,1]).

E-mail address: saltinok43@yahoo.com.

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The main result of this paper is a verification of a formula for the least element of the set of divisors $X \in \mathcal{E}^+(E)$ greater than or equal to $Y + N$ for a given $Y \in \mathcal{E}^+(E)$ and a non-Tjurina component N of Y (see Theorem 2.5). This formula was stated in [8] but no proof has been given. In the Tjurina case the corresponding formula is given by Pinkham (see Theorem 2.3). It follows that starting with the fundamental cycle of E any element $D \in \mathcal{E}^+(E)$ can be obtained by applying iteratively Pinkham's theorem and Theorem 2.5. It is important to observe that the two operations in this process are necessary. Related definitions and notations are given below.

2. Constructing elements of semigroup $\mathcal{E}^+(E)$

Let (X, ξ) be a germ of a normal complex analytic surface having a singularity at ξ. For a fixed resolution $\pi : \widetilde{X} \to X$ of the singularity (X, ξ) the exceptional fiber $E = \pi^{-1}(x)$ is connected and of dimension 1. Let $E = \bigcup E_i$ where $E_1, \dots, E_n$ are the irreducible components of E.

Let $\mathcal{E}^+(E)$ be the set of non-zero effective divisors $Y = \sum_{i=0}^n a_i E_i$ such that $Y \cdot E_i \leqslant 0$ for all i. This set is non-empty (see Zariski [9]). It is also a semigroup under addition: $D_1 + D_2 = \sum_{i=0}^n (n_i + m_i) E_i$ where $D_1 = \sum n_i E_i$ and $D_2 = \sum m_i E_i \in \mathcal{E}^+(E)$. Since E is connected, $a_i \geqslant 1$ for all $i = 1, \dots, n$.

For a given $Y = \sum_{i=1}^n a_i E_i \in \mathcal{E}^+(E)$ we define the multiplicity of E_i in Y as the coefficient a_i of E_i in Y and denote it by $\operatorname{mult}_Y E_i$. There is a partial ordering on $\mathcal{E}^+(E)$ defined by $Y \leqslant Y'$ if $\operatorname{mult}_Y E_i \leqslant \operatorname{mult}_{Y'} E_i$ for all i. For any $Y, Y' \in \mathcal{E}^+(E)$ we can define $\min(Y, Y') = \sum \min(a_i, a'_i) E_i$, which is again in $\mathcal{E}^+(E)$. Therefore, there exists the absolutely minimal effective cycle in $\mathcal{E}^+(E)$, called the fundamental cycle Z of E. It can be determined by Laufer's algorithm (see Laufer [4]). This algorithm may be extended in the following way:

Lemma 2.1. *Let X be an effective cycle supported on E. There is a cycle $Z_1 \geqslant X$ such that $Z_1 \in \mathcal{E}^+(E)$. The smallest element in $\mathcal{E}^+(E)$ greater than or equal to X is given by Laufer's algorithm.*

Proof. Let $Z = \sum a_i E_i$ be the fundamental cycle and $X = \sum b_i E_i$ an effective cycle. There exists $n \in \mathbb{N}$ such that $X \leqslant nZ$. Since $nZ \in \mathcal{E}^+(E)$, this proves the first part of the lemma. Denote by Y the absolute minimum of the set of elements $\mathcal{E}^+(E)$ greater than or equal to X.

For the construction of Y, we start with the effective cycle X and, assuming that $X \notin \mathcal{E}^+(E)$, add to it any irreducible component E_{i_0} such that $X \cdot E_{i_0} > 0$. Setting $X_1 = X + E_{i_0}$, one shows that $X_1 \leqslant Y$. Iterating this step we construct a sequence $X_1, X_2, \dots$ of effective cycles supported by E such that if there is an irreducible component E_{i_j} for which $X_j \cdot E_{i_j} > 0$ we set $X_{j+1} = X_j + E_{i_j}$. Eventually, we stop when $X_l \cdot E_j \leqslant 0$ for all j. This process is finite because we necessarily have $X_j \leqslant Y$ for all j. By definition, we have also $Y \leqslant X_l$, therefore $X_l = Y$. Hence $X_l \in \mathcal{E}^+(E)$ is the smallest one. □

From now on, we assume that the singularity is rational.

Definition 2.2. A Tjurina component of $Y \in \mathcal{E}^+(E)$ is a maximal connected set T of irreducible components of E such that $Y \cdot E_i = 0$ for all irreducible components E_i in T. Non-Tjurina components of Y are irreducible components E_i of E such that $Y \cdot E_i < 0$.

We let $X = Y + E_i$, where $Y \in \mathcal{E}^+(E)$ and E_i is any irreducible component of E. Our aim is to find out combinatorially what the least element Z_1 in Lemma 2.1 should be. We can divide the problem in two parts. First, if E_i is contained in a Tjurina component of Y then this question is answered by Pinkham (see Theorem 2.3). Second, if E_i is a non-Tjurina component of Y we answer it in Theorem 2.5.

Theorem 2.3 *(Pinkham's Theorem [7]). Let E_i be an irreducible component of E which is contained in a Tjurina component T of Z. Then the least element $D \geqslant Z + E_i$ of $\mathcal{E}^+(E)$ is equal to $Z + Z(T)$, where $Z(T)$ is the fundamental cycle of T.*

We note that Pinkham's theorem still holds if the fundamental cycle Z is replaced by any $Y \in \mathcal{E}^+(E)$.

Definition 2.4. Let us denote by T a Tjurina component of $Y \in \mathcal{E}^+(E)$ and by C an irreducible component of E which is contained in T. Let $T^0 = T$ and $Z(T^0)$ be the fundamental cycle of T^0. If $Z(T^0) \cdot C < 0$ then define the depth of C in T^0 to be 0. If $Z(T^0) \cdot C = 0$, then let T^1 denote the Tjurina component of $Z(T^0)$ which contains C. In this way, one can form a finite chain $T^l \subset T^{l-1} \subset \cdots \subset T^1 \subset T^0$ such that T^i is the Tjurina component of the fundamental cycle $Z(T^{i-1})$ containing C for $i = 1, \ldots, l$, where $Z(T^0) \cdot C = Z(T^1) \cdot C = \cdots = Z(T^{l-1}) \cdot C = 0$ and $Z(T^l) \cdot C < 0$. The length of this chain, l, is called the desingularization depth of C and is denoted by $\operatorname{depth}_T C$.

Theorem 2.5. *Let F be a non-Tjurina component of $Y \in \mathcal{E}^+(E)$ attached to Tjurina components T_i for $i = 1, \ldots, q$ and C_i the unique irreducible component of T_i such that $C_i \cap F \neq \emptyset$. Let $A_F = \{D \in \mathcal{E}^+ \mid D \geqslant Y + F\}$. Then the least element in A_F is given by*

$$Y + F + \sum_{i=1}^{q} \sum_{j=0}^{l_i} Z\big(T_i^j\big)$$

where $l_i = \operatorname{depth}_{T_i} C_i$. In particular, if F is a non-Tjurina component of Y not attached to any Tjurina ones then the least element of A_F is $Y + F$.

To prove Theorem 2.5 we need the following lemma:

Lemma 2.6. *Let F be a non-Tjurina component of Z. Let denote by $T_1, \ldots, T_r$ the Tjurina components of Z and by $C_1, \ldots, C_r$ the irreducible components of $T_1, \ldots, T_r$ respectively such that $C_t \cap F \neq \emptyset$ for all t where $t = 1, \ldots, r$. Then*

$$(r-1) + \sum \operatorname{depth}_{T_i} C_i \leqslant -F^2.$$

Proof. We will only prove this lemma for $r = 1$. The proof can easily be generalized for any r. So, let $T = T_1$ be the unique Tjurina component attached to F. Start with $Z_0 = F$ and apply Laufer's algorithm within T_1 to get a sequence $Z_0, Z_1 = F + C_1, \ldots, Z_{t_0} = F + Z(T)$ for some $t_0 > 0$. Either $Z(T) \cdot C_1$ is equal to zero or less than zero. If it is zero, then apply Laufer's algorithm within the Tjurina component T^1 of $Z(T)$ containing C_1 to get a sequence $Z_{t_0}, Z_{t_0+1} = Z_{t_0} + C_1, \ldots, Z_{t_1} = Z_{t_0} + Z(T^1)$ for some $t_1 > t_0$. If $Z(T^1) \cdot C_1 = 0$, then we repeat the process to reach $Z_{t_2} = Z_{t_1} + Z(T^2)$ for some $t_2 > t_1$, where T^2 is the Tjurina component of $Z(T^1)$ containing C_1. Continuing in this way we will eventually obtain $Z_{t_l} = F + Z(T^0) + Z(T^1) + Z(T^2) + \cdots + Z(T^l)$ for some $t_l > t_{l-1}$, where $T^0 = T$ and by induction T^l is the Tjurina component of $Z(T^{l-1})$ which contains C_1 and C_1 is the non-Tjurina component of $Z(T^l)$. Then $Z_{t_l} \cdot F = F^2 + (l+1) \leqslant 1$ by Laufer's algorithm. This implies that $l \leqslant -F^2$, where $l = \operatorname{depth}_T C_1$. $\square$

Proof of Theorem 2.5. The proof is given for the case $q = 1$. The general case is similar. Let F be a non-Tjurina component of $Y \in \mathcal{E}^+$ attached to a single Tjurina component T and C_1 be the unique irreducible component of T such that $C_1 \cap F \neq \emptyset$. We will prove that the least element in A_F is given by $Y + F + \sum_{i=0}^{l_1} Z(T^i)$, where $l_1 = \operatorname{depth}_T C_1$.

Let us denote $Y + F + \overline{Z}$ by $\overline{Y}$ where $\overline{Z} = \sum_{i=0}^{l_1} Z(T^i)$. We divide the proof into two parts. The first part is to prove that $\overline{Y} \in \mathcal{E}^+$. The second part is to show that every element of A_F contains $\overline{Y}$.

For the first part, we check via a case-by-case analysis that $\overline{Y} \cdot C \leqslant 0$ for all irreducible components C of E.

Case (a) $C = F$. By Lemma 2.6 together with $Y \cdot F < 0$, we have $\overline{Y} \cdot F = Y \cdot F + F^2 + (l_1 + 1) \leqslant 0$.

Case (b) $C = C_1$. By definition of the desingularization depth of C_1 together with $Y \cdot C_1 = 0$ and $F \cdot C_1 = 1$, we have $\overline{Y} \cdot C_1 = Y \cdot C_1 + F \cdot C_1 + \sum_{i=0}^{l_1} Z(T^i) \cdot C_1 \leqslant 0$.

Case (c) C is contained in T and $F \cdot C = 0$. It is sufficient to prove that $\sum_{i=0}^{l_1} Z(T^i) \cdot C \leqslant 0$. We find the first index l, if there exists, such that $Z(T^l) \cdot C < 0$. This implies that $C \notin Z(T^{l+1})$. Hence $Z(T^{l+1}) \cdot C = 0$ or 1. If $Z(T^{l+1}) \cdot C = 0$ then $Z(T^i) \cdot C = 0$ for $i = l+2, \ldots, l_1$. Therefore, $\overline{Z} \cdot C < 0$. Now we assume that $Z(T^{l+1}) \cdot C = 1$. We consider the subtree $F \cup Z(T^{l+1}) \cup C$ of Γ, the dual graph of E, and apply Laufer's algorithm to get a sequence:

$$Z_0 = F, Z_1 = F + C_1, \ldots, Z_t = F + Z(T^{l+1}) + \cdots + Z(T^{l_1}), Z_{t+1} = Z_t + C$$

where $Z_t \cdot C \leqslant 1$. This gives that $\sum_{i=l+1}^{l_1} Z(T^i) \cdot C \leqslant 1$. Hence $\overline{Y} \cdot C = (Y + F + \sum_{i=0}^{l-1} Z(T^i)) \cdot C + Z(T^l) \cdot C + (\sum_{i=l+1}^{l_1} Z(T^i)) \cdot C \leqslant 0$.

Case (d) C is not contained in T and $C \cap T = \emptyset$. Hence $\sum_{i=0}^{l_1} Z(T^i) \cdot C = 0$. If $F \cdot C = 1$ then C is non-Tjurina and hence $\overline{Y} \cdot C = Y \cdot C + F \cdot C \leqslant 0$. If $F \cdot C = 0$ then $\overline{Y} \cdot C = Y \cdot C \leqslant 0$.

Case (e) C is not contained in T and $C \cap T \neq \emptyset$. This implies that there is an i such that $C_i \in T$ and $C_i \cdot C = 1$, $Y \cdot C < 0$ and $F \cdot C = 0$. Now, we can consider the subtree $F \cup T \cup C$ of Γ and apply Laufer's algorithm in order to get a sequence:

$$Z_0 = F, Z_1 = F + C_1, \ldots, Z_t = F + \sum_{j=0}^{l_1} Z(T^j), Z_{t+1} = Z_t + C,$$

where $Z_t \cdot C \leqslant 1$. Since $Y \cdot C < 0$ and $Z_t = \overline{Z}$, we have $\overline{Y} \cdot C = (Y + Z_t) \cdot C \leqslant 0$.

For the second part, by Laufer's algorithm any D in A_F contains $Y + F + C_1$ and hence also $Y + F + \sum_{i=0}^{l_1} Z(T^i)$. $\square$

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