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Dynamical Systems

# Polycyclic groups of diffeomorphisms of the closed interval

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**Abstract**

We give a classification of polycyclic groups of orientation-preserving $C^2$-diffeomorphisms of the closed interval. This shows that many polycyclic groups of $C^2$-diffeomorphisms of the half-open interval are not the restriction of groups of $C^2$-diffeomorphisms of the closed interval. ***To cite this article: Y. Matsuda, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


**Résumé**

**Groupes polycycliques de difféomorphismes de l'intervalle fermé.** On donne une classification des groupes polycycliques de difféomorphismes directs et de classe $C^2$ de l'intervalle fermé. Cela montre que il y a des groupes polycycliques de difféomorphsmes de classe $C^2$ de l'intervalle demi-ouvert qui ne sont pas des restrictions des groupes de difféomorphsmes de classe $C^2$ de l'intervalle fermée. ***Pour citer cet article : Y. Matsuda, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


## 1. Introduction

The purpose of this note is to describe a contrast between the groups of diffeomorphisms of the half-open interval and those of the closed interval. Precisely, we compare the group $\mathrm{Diff}^2_+([0,1])$ of orientation-preserving $C^2$-diffeomorphisms of the closed interval $[0,1]$ with the group $\mathrm{Diff}^2([0,1[)$ of $C^2$-diffeomorphisms of the half-open interval $[0,1[$. The former can be regarded as a subgroup of the later by considering the restriction to the half-open interval. We focus on polycyclic subgroups of these two groups.

Recall that a group $\Gamma$ is said to be *polycyclic* if there exists a finite sequence of subgroups

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = \{1\}$$

such that for every $i = 1, \dots, n$, $\Gamma_i$ is normal in $\Gamma_{i-1}$ and $\Gamma_{i-1}/\Gamma_i$ is cyclic. It follows from the definition that every polycyclic group is solvable. On the other hand, it is known that a solvable group is polycyclic if and only if each of its subgroup is finitely generated (see [13], pp. 432–433, Proposition 4.1). Therefore every polycyclic group $\Gamma$ has

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a unique non-trivial nilpotent normal subgroup $N$ which is maximal with respect to inclusion. This subgroup $N$ is called the *nilradical* of $\Gamma$ (see Chapter IV of [11], pp. 56–58 for the details).

After the works of Kopell [3] and Plante–Thurston [10] on abelian and nilpotent subgroups of $\mathrm{Diff}^2([0,1[)$ respectively, Plante studied solvable subgroups of $\mathrm{Diff}^2([0,1[)$ in [8] and [9]. Examples of non-nilpotent polycyclic groups are obtained as subgroups of the group $\mathrm{Aff}_+(\mathbb{R})$ of orientation-preserving affine transformations of the real line $\mathbb{R}$. Based on this fact, Plante constructed polycyclic subgroups of $\mathrm{Diff}^2([0,1[)$ by choosing a diffeomorphism $\varphi$ from the real line $\mathbb{R}$ onto the open-interval $]0,1[$ such that the conjugates of elements of $\mathrm{Aff}_+(\mathbb{R})$ by $\varphi$ are extended to $C^2$-diffeomorphisms on the half-open interval $[0,1[$ (see [9], p. 48). By an appropriate choice of the diffeomorphism $\varphi$, these conjugates are extended to $C^2$-diffeomorphisms on the closed interval $[0,1]$. Thus the affine group $\mathrm{Aff}_+(\mathbb{R})$ can be regarded as a subgroup of $\mathrm{Diff}^2_+([0,1]) \subset \mathrm{Diff}^2([0,1[)$ and we obtain polycyclic subgroups of $\mathrm{Diff}^2_+([0,1])$. For the details and another construction by using the action of $\mathrm{Aff}_+(\mathbb{R})$ on the circle, see [12], pp. 231–232.

Plante also gave the following partial classification of polycyclic subgroups of $\mathrm{Diff}^2([0,1[)$ based on Corollary 4.6 in [8] (see [9], p. 48, Theorem B):

**Theorem** *(Plante). Let $\Gamma$ be a polycyclic subgroup of* $\mathrm{Diff}^2([0,1[)$. *If the nilradical of $\Gamma$ has no global fixed points in* $]0,1[$*, then $\Gamma$ is topologically conjugate to a subgroup of* $\mathrm{Aff}_+(\mathbb{R})$.

Note that in the original statement of Theorem B in [9] Plante assumed only that $\Gamma$ has no global fixed points in $]0,1[$ and failed to write the assumption on the fixed point set of the nilradical though his argument in the proof shows the above theorem.

Later Moriyama [5] showed that the assumption on the fixed point set of the nilradical in Plante's theorem is necessary. In fact, he constructed polycyclic subgroups of $\mathrm{Diff}^2([0,1[)$ such that they have no global fixed points in $]0,1[$ and their restrictions to $]0,1[$ do not conjugate to subgroups of $\mathrm{Aff}_+(\mathbb{R})$ (see [5], p. 415). He also classified polycyclic subgroups of $\mathrm{Diff}^2([0,1[)$ into two types in accordance with the dynamics of the nilradical (see [5], p. 399). His ideas were pursued by Navas [6,7] to give a classification of solvable subgroups of $\mathrm{Diff}^2([0,1[)$.

In this paper we show that a different situation arises when we restrict ourselves to $\mathrm{Diff}^2_+([0,1])$. In fact, when we replace $\mathrm{Diff}^2([0,1[)$ by $\mathrm{Diff}^2_+([0,1])$ in Plante's theorem, the assumption on the nilradical must necessarily hold if the group $\Gamma$ has no global fixed point in $]0,1[$. Our main result is the following:

**Theorem 1.1.** *Let $\Gamma$ be a polycyclic subgroup of* $\mathrm{Diff}^2_+([0,1])$. *If $\Gamma$ has no global fixed points in* $]0,1[$*, then it is topologically conjugate to a subgroup of* $\mathrm{Aff}_+(\mathbb{R})$.

This shows that Moriyama's polycyclic subgroups of $\mathrm{Diff}^2([0,1[)$ are not restrictions of polycyclic subgroups of $\mathrm{Diff}^2_+([0,1])$. Note that Theorem 1.1 does not hold true if we replace $\Gamma$ by a solvable (but not polycyclic) subgroup of $\mathrm{Diff}^2_+([0,1])$ (see Remark 2).

The key ingredient of the proof of Theorem 1.1 is a variation of Kopell's lemma for $\mathrm{Diff}^2_+([0,1])$ due to S. Druck and S. Firmo (see Theorem 2.1).

## 2. A variation of Kopell's lemma

We begin by recalling Kopell's lemma ([3], Lemma 1).

**Lemma** *(Kopell's lemma). Let $f$ and $g$ be commuting elements of* $\mathrm{Diff}^2([0,1[)$. *If $f$ fixes no points in* $]0,1[$ *and $g$ fixes a point in* $]0,1[$*, then $g$ is the identity.*

Note that in this lemma the group generated by $g$ is a normal subgroup of the group generated by $f$ and $g$. This observation results in the following fact:

*Let $\Gamma$ be a subgroup of* $\mathrm{Diff}^2([0,1[)$. *If $N$ is an infinite cyclic normal subgroup of $\Gamma$, then we have* $\partial\,\mathrm{Fix}(N) \subset \mathrm{Fix}(\Gamma)$.

Druck and Firmo ([1], Theorem 5.3) pointed out that when we restrict ourselves to $\mathrm{Diff}^2_+([0,1])$ this fact can be extended as follows:

**Theorem 2.1** *(Druck and Firmo). Let $\Gamma$ be a subgroup of* $\mathrm{Diff}^2_+([0,1])$. *If $N$ is a finitely generated abelian normal subgroup of $\Gamma$, then we have* $\partial\,\mathrm{Fix}(N) \subset \mathrm{Fix}(\Gamma)$.

**Remark 1.** Theorem 2.1 still holds true even if we replace $\mathrm{Diff}^2_+([0,1])$ by $\mathrm{Diff}^{1+bv}_+([0,1])$ and we do not assume that $N$ is abelian (the proof will be given in [4]). However, this extension is unnecessary for the proof of Theorem 1.1 and we do not go further here.

### 3. Proof of Theorem 1.1

Let $\Gamma$ be a polycyclic subgroup of $\mathrm{Diff}^2_+([0,1])$ and assume that $\mathrm{Fix}(\Gamma) = \{0,1\}$. We denote $N$ the nilradical of $\Gamma$. Then $N$ is a finitely generated nilpotent normal subgroup of $\Gamma$. Since $N$ is nilpotent, it follows from the theorem of Plante and Thurston (see [10], Theorem 4.5) that $N$ is abelian. Hence it follows from Theorem 2.1 that $N$ has no global fixed points in $]0,1[$. Then by Plante's theorem the group $\Gamma$ is topologically conjugate to a subgroup of $\mathrm{Aff}_+(\mathbb{R})$. Thus we have finished the proof of Theorem 1.1.

**Remark 2.** Theorem 1.1 does not hold true if we replace $\Gamma$ by a solvable subgroup of $\mathrm{Diff}^2_+([0,1])$. To see this, we quote an example from Godbillon's book (see [2], Chapitre V, p. 315, Exercices 1.7 vi)).

Let $f$ be an element of $\mathrm{Diff}^2_+([0,1])$ such that $f(x) > x$ for all $f \in\, ]0,1[$. We take a point $a \in\, ]0,1[$ and let $g$ be an element of $\mathrm{Diff}^2_+([0,1])$ such that $\mathrm{Fix}(g) = [0, f(a)] \cup [a,1]$. We denote $\Gamma$ the subgroup of $\mathrm{Diff}^2_+([0,1])$ generated by $f$ and $g$ and $N$ the normal closure of $\{g\}$ in $\Gamma$. We can see that $N$ is generated by $\{f^{-j} g f^j\colon\ j \in \mathbb{Z}\}$ and the quotient group $\Gamma/N$ is isomorphic to the infinite cyclic group generated by $f$. Then it follows that $N$ is a free abelian normal subgroup of $\Gamma$ which is of infinite rank and the group $\Gamma$ is solvable. Moreover we have $\mathrm{Fix}(\Gamma) = \{0,1\}$ and $\mathrm{Fix}(N) = \partial\,\mathrm{Fix}(N) = \{f^j(a)\colon\ j \in \mathbb{Z}\} \cup \{0,1\}$.

This shows that Theorem 1.1 does not hold true if we replace $\Gamma$ by a solvable subgroup of $\mathrm{Diff}^2_+([0,1])$. This also shows that Theorem 2.1 does not hold true if we replace $N$ by an infinitely generated abelian normal subgroup of $\Gamma$.

**Remark 3.** As pointed out by the referee, Theorem 1.1 still holds true when we replace $\mathrm{Diff}^2_+([0,1])$ by the group $\mathrm{PL}_+([0,1])$ of orientation-preserving piecewise linear homeomorphisms of the closed interval $[0,1]$. Moreover we can show the following: *Let $\Gamma$ be a polycyclic subgroup of* $\mathrm{PL}_+([0,1])$. *If $\Gamma$ has no global fixed points in* $]0,1[$*, then it is cyclic*. The proof involves a few facts peculiar to groups of piecewise linear homeomorphisms of the interval and will be given in a forthcoming paper.

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