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Homological Algebra

Homology with coefficients of Leibniz n-algebras

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Abstract

Co-representations of Leibniz n-algebras are defined as left modules over the universal enveloping algebra. We define the homology of a Leibniz n-algebra L with coefficients in a co-representation M as the homology of the Leibniz complex of $\mathrm{L}^{\otimes n-1}$ over the co-representation $\mathrm{M} \otimes \mathrm{L}$.

We prove the cancellation of the homology over free objects and the generalization of the following isomorphism in Leibniz homology $HL_\star(\mathrm{L}, \mathrm{L}) \cong HL_{\star+1}(\mathrm{L}, K)$ from Leibniz algebras to Leibniz n-algebras. ***To cite this article: J.M. Casas, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Homologie avec des coefficients des n-algèbres de Leibniz. Les co-représentations des n-algèbres de Leibniz sont définies comme les modules à gauche sur l'algèbre enveloppante universelle. Nous définissons l'homologie de la n-algèbre de Leibniz L à coefficients dans une co-représentation M comme l'homologie du complexe de Leibniz de $\mathrm{L}^{\otimes n-1}$ sur la co-représentation $\mathrm{M} \otimes \mathrm{L}$.

Nous démontrons l'annulation de l'homologie sur les objets libres et nous généralisons l'isomorphisme $HL_\star(\mathrm{L}, \mathrm{L}) \cong HL_{\star+1}(\mathrm{L}, K)$ des algèbres de Leibniz aux n-algèbres de Leibniz. ***Pour citer cet article : J.M. Casas, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let K be a field. A *Leibniz n-algebra* [3] L is a K-vector space equipped with an n-linear bracket $[-, \dots, -]$: $\mathrm{L}^{\otimes n} \to \mathrm{L}$ satisfying the following fundamental identity

$$\big[[x_1, \dots, x_n], y_1, \dots, y_{n-1}\big] = \sum_{i=1}^{n} \big[x_1, \dots, x_{i-1}, [x_i, y_1, \dots, y_{n-1}], x_{i+1}, \dots, x_n\big]. \tag{1}$$

In the case $n = 2$ the fundamental identity (1) becomes the Leibniz identity, so a Leibniz 2-algebra is just a Leibniz algebra [7].

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Representations were introduced as cohomology coefficients in [3] and it was proven in [2] that the category of representations of a Leibniz n-algebra L is equivalent to the category of right modules over the universal enveloping algebra $\mathtt{U}_n\mathtt{L}(\mathrm{L}) = \mathrm{T}((\mathrm{L}^{\otimes(n-1)})^l \oplus (\mathrm{L}^{\otimes(n-1)})^{1m} \oplus \overset{n-2}{\cdots} \oplus (\mathrm{L}^{\otimes(n-1)})^{n-2m} \oplus (\mathrm{L}^{\otimes(n-1)})^r)/\, I$ where I is the n-sided ideal generated by the corresponding relations (see [2] for details).

2. Co-representations

Definition 2.1. A co-representation of a Leibniz n-algebra L is a K-vector space M equipped with n actions

$$[-, \overset{n}{\ldots}, -] : \mathrm{L}^{\otimes i} \otimes \mathrm{M} \otimes \mathrm{L}^{\otimes n-i-1} \to \mathrm{M}, \quad 0 \leqslant i \leqslant n-1,$$

satisfying the following $(2n-1)$ axioms:
1. $\lambda_i([l_1, \ldots, l_n], l_{n+1}, \ldots, l_{2n-2}) = \sum_{j=1}^n \lambda_i(l_j, l_{n+1}, \ldots, l_{2n-2}) \cdot \lambda_{n+1-j}(l_1, \ldots, \widehat{l_j}, \ldots, l_n)$;
2. $\lambda_k[l_1 \otimes \cdots \otimes l_{n-1}, l_n \otimes \cdots \otimes l_{2n-2}] = \lambda_k(l_1, \ldots, l_{n-1}) \cdot \lambda_n(l_n, \ldots, l_{2n-2}) - \lambda_n(l_n, \ldots, l_{2n-2}) \cdot \lambda_k(l_1, \ldots, l_{n-1})$
for $1 \leqslant i \leqslant n-1$ and $1 \leqslant k \leqslant n$, where the multilinear applications $\lambda_i : \mathrm{L}^{\otimes n-1} \to \mathrm{End}_K(\mathrm{M})$, $1 \leqslant i \leqslant n$, are defined by $\lambda_i(l_1, \ldots, l_{n-1})(m) := [l_1, \ldots, l_{i-1}, m, l_i, \ldots, l_{n-1}]$.

Theorem 2.2. *The category of co-representations over a Leibniz n-algebra* L *is equivalent to the category of left modules on the universal enveloping algebra* $\mathtt{U}_\mathtt{n}\mathtt{L}(\mathrm{L})$.

Theorem 2.3. *If* M *is a representation of the Leibniz n-algebra* L, *then* $\mathrm{M} \otimes \mathrm{L}^{\otimes n-2}$ *is a* L*-co-representation with respect to the actions*

$$[-, \ldots, -] : \overbrace{\mathrm{L} \otimes \cdots \otimes \mathrm{L}}^{n-1} \otimes \big(\mathrm{M} \otimes \mathrm{L}^{\otimes n-2}\big) \to \mathrm{M} \otimes \mathrm{L}^{\otimes n-2},$$
$$\big[l_1, \ldots, l_{n-1}, m \otimes l'_1 \otimes \cdots \otimes l'_{n-2}\big] = -m \otimes l'_1 \otimes \cdots \otimes l'_{n-3} \otimes \big[l'_{n-2}, l_1, \ldots, l_{n-1}\big] - m \otimes l'_1 \otimes \cdots \otimes l'_{n-4} \otimes \big[l'_{n-3}, l_1, \ldots, l_{n-1}\big] \otimes l'_{n-2} - \cdots - m \otimes \big[l'_1, l_1, \ldots, l_{n-1}\big] \otimes l'_2 \otimes \cdots \otimes l'_{n-2} - [m, l_1, \ldots, l_{n-1}] \otimes l'_1 \otimes \cdots \otimes l'_{n-2}.$$

For $2 \leqslant i \leqslant n-1$

$$[-, \ldots, -] : \overbrace{\mathrm{L} \otimes \cdots \otimes \mathrm{L}}^{i-1} \otimes \big(\mathrm{M} \otimes \mathrm{L}^{\otimes n-2}\big) \otimes \overbrace{\mathrm{L} \otimes \cdots \otimes \mathrm{L}}^{n-i} \to \mathrm{M} \otimes \mathrm{L}^{\otimes n-2},$$
$$\big[l_1, \ldots, l_{i-1}, m \otimes l'_1 \otimes \cdots \otimes l'_{n-2}, l_i, \ldots, l_{n-1}\big] = 0,$$
$$[-, \ldots, -] : \big(\mathrm{M} \otimes \mathrm{L}^{\otimes n-2}\big) \otimes \overbrace{\mathrm{L} \otimes \cdots \otimes \mathrm{L}}^{n-1} \to \big(\mathrm{M} \otimes \mathrm{L}^{\otimes n-2}\big),$$
$$\big[m \otimes l'_1 \otimes \cdots \otimes l'_{n-2}, l_1, \ldots, l_{n-1}\big] = \big[m, l'_1, \ldots, l'_{n-2}, l_1\big] \otimes l_2 \otimes \cdots \otimes l_{n-1}.$$

Remark 1. Theorem 2.3 in the case $n = 2$ says that a representation of a Leibniz algebra L can be endowed with a structure of L-co-representation with respect to the actions

$$\{-, -\} : \mathrm{L} \otimes M \to M, \quad l \otimes m \mapsto \{l, m\} = -[m, l],$$
$$\{-, -\} : M \otimes \mathrm{L} \to M, \quad m \otimes l \mapsto \{m, l\} = [m, l]$$

which easily follows from [8].

On the other hand, Corollary 1.4 in [6] establishes the equivalence between the categories of representations and co-representations of Leibniz algebras. Nevertheless this result does not hold for Leibniz n-algebras $n \geqslant 3$, as the following counterexample shows:

Let L be the 2-dimensional Leibniz 3-algebra with basis $\{e_1, e_2\}$ and multiplication table given by $[e_2, e_1, e_1] = [e_2, e_1, e_2] = [e_2, e_2, e_1] = [e_2, e_2, e_2] = e_1 - e_2$ and zero otherwise. After a tedious but straightforward checking we can see that L is an L-representation, but it is not an L-co-representation.

3. Homology with coefficients

Proposition 3.1. *Let* L *be a Leibniz n-algebra and let* M *be a co-representation of* L. *Then* $\mathrm{M}\otimes\mathrm{L}$ *is a co-representation* [8] *over the Leibniz algebra* $\mathrm{L}^{\otimes n-1}$ *with bracket* $[x_1\otimes\cdots\otimes x_{n-1}, y_1\otimes\cdots\otimes y_{n-1}]=\sum_{i=1}^{n-1}[x_1,\dots,[x_i,y_1,\dots,y_{n-1}],\dots,x_n]$.

Proof. The following maps give the actions

$$[-,-]:\mathrm{L}^{\otimes n-1}\otimes(\mathrm{M}\otimes\mathrm{L})\to\mathrm{M}\otimes\mathrm{L},$$
$$[l_1\otimes\cdots\otimes l_{n-1}, m\otimes l_n]:=[l_1,\dots,l_{n-1},m]\otimes l_n-m\otimes[l_n,l_1,\dots,l_{n-1}],$$
$$[-,-]:(\mathrm{M}\otimes\mathrm{L})\otimes\mathrm{L}^{\otimes n-1}\to\mathrm{M}\otimes\mathrm{L},$$
$$[m\otimes l_n, l_1\otimes\cdots\otimes l_{n-1}]:=m\otimes[l_n,l_1,\dots,l_{n-1}]-[l_1,\dots,l_{n-1},m]\otimes l_n-[m,l_n,l_1,\dots,l_{n-2}]\otimes l_{n-1}$$
$$-[l_n,m,l_2,\dots,l_{n-1}]\otimes l_1-[l_n,l_1,m,l_3,\dots,l_{n-1}]\otimes l_2-\cdots-[l_n,l_1,\dots,l_{n-3},m,l_{n-1}]\otimes l_{n-2}.\qquad\square$$

Let L be a Leibniz n-algebra and let M be a co-representation of L. We define the chain complex

$$_nCL_*(\mathrm{L},\mathrm{M}):=CL_*\big(\mathrm{L}^{\otimes n-1},\mathrm{M}\otimes\mathrm{L}\big)\tag{2}$$

where $CL_\star$ denotes the Leibniz complex in [8]. We define the homology of L with coefficients in M as the homology of the Leibniz complex (2). Thus, by definition, we have that

$$_nHL_*(\mathrm{L},\mathrm{M}):=HL_*\big(\mathrm{L}^{\otimes n-1},\mathrm{M}\otimes\mathrm{L}\big).$$

Let us observe that in the case $n=2$ we obtain

$$_2CL_k(\mathrm{L},\mathrm{M})=CL_k(\mathrm{L},\mathrm{M}\otimes\mathrm{L})=(\mathrm{M}\otimes\mathrm{L})\otimes\mathrm{L}^k=\mathrm{M}\otimes\mathrm{L}^{k+1}=CL_{k+1}(\mathrm{L},\mathrm{M})$$

thus

$$_2HL_k(\mathrm{L},\mathrm{M})=HL_{k+1}(\mathrm{L},\mathrm{M}).$$

On the other hand, if we consider M = K as a trivial co-representation of L, then $_nHL_k(\mathrm{L},\mathrm{M})$ coincides with the homology with trivial coefficients given in [1].

If M is a trivial co-representation of L, then $_nHL_0(\mathrm{L},\mathrm{M})\cong\mathrm{M}\otimes\mathrm{L}/[\mathrm{L},\overset{n}{\dots},\mathrm{L}]$.

In the case $n=2$ we recover Proposition 2 b) in [4] since $HL_1(\mathrm{L},\mathrm{M})\cong{}_2HL_0(\mathrm{L},\mathrm{M})\cong\mathrm{M}\otimes\mathrm{L}/[\mathrm{L},\mathrm{L}]$.

In case of the trivial co-representation $\mathrm{M}=K$, $_nHL_0(\mathrm{L},K)\cong\mathrm{L}/[\mathrm{L},\overset{n}{\dots},\mathrm{L}]=\mathrm{L}_{\mathrm{ab}}$ (see [1]).

Proposition 3.2. *Let* L *be a free Leibniz n-algebra and let* M *be a co-representation of* L. *Then*

$$_nHL_k(\mathrm{L},\mathrm{M})=0,\quad k\geqslant 1.$$

Proof. By Remark 4.9 in [3] we have that $\mathrm{L}^{\otimes n-1}$ is a free Leibniz algebra. Thanks to Corollary 3.5 in [8] we have that $HL_k(\mathrm{L}^{\otimes n-1},-)=0$ for $k\geqslant 2$. Thus $_nHL_k(\mathrm{L},\mathrm{M})=0$, for $k\geqslant 1$. $\square$

Lemma 3.3. *Let* L *be a Leibniz n-algebra, then the underlying* K*-vector space* $\mathrm{L}^{\otimes n-1}$ *is endowed with a structure of co-representation over* L *by means of the following operations*:

$$[-,\dots,-]:\overbrace{\mathrm{L}\otimes\cdots\otimes\mathrm{L}}^{n-1}\otimes\mathrm{L}^{\otimes n-1}\to\mathrm{L}^{\otimes n-1},$$
$$[l_1,\dots,l_{n-1},l'_1\otimes\cdots\otimes l'_{n-1}]=-l'_1\otimes\cdots\otimes l'_{n-2}\otimes[l'_{n-1},l_1,\dots,l_{n-1}]$$
$$-l'_1\otimes\cdots\otimes l'_{n-3}\otimes[l'_{n-2},l_1,\dots,l_{n-1}]\otimes l'_{n-1}-\cdots-[l'_1,l_1,\dots,l_{n-1}]\otimes l'_2\otimes\cdots\otimes l'_{n-1}.$$

For $2\leqslant i\leqslant n-1$,

$$[-,\dots,-]:\overbrace{\mathrm{L}\otimes\cdots\otimes\mathrm{L}}^{i-1}\otimes\mathrm{L}^{\otimes n-1}\otimes\overbrace{\mathrm{L}\otimes\cdots\otimes\mathrm{L}}^{n-i}\to\mathrm{L}^{\otimes n-1},$$
$$[l_1,\dots,l_{i-1},l'_1\otimes\cdots\otimes l'_{n-1},l_i\otimes\cdots\otimes l_{n-1}]=0,$$
$$[-,\dots,-]:\mathrm{L}^{\otimes n-1}\otimes\overbrace{\mathrm{L}\cdots\otimes\mathrm{L}}^{n-1}\to\mathrm{L}^{\otimes n-1},$$
$$[l'_1\otimes\cdots\otimes l'_{n-1},l_1,\dots,l_{n-1}]=[l'_1,\dots,l'_{n-1},l_1]\otimes l_2\otimes\cdots\otimes l_{n-1}.$$

Proof. Take $\mathrm{M}=\mathrm{L}$ in Theorem 2.3. □

Proposition 3.4. *Let* L *be a Leibniz* n*-algebra, then*

$$_nHL_k(\mathrm{L},\mathrm{L}^{\otimes n-1})\cong {_nHL_{k+1}}(\mathrm{L},K).$$

Proof. We compute:

$$\begin{aligned}_nCL_k(\mathrm{L},\mathrm{L}^{\otimes n-1})&\cong CL_k(\mathrm{L}^{\otimes n-1},\mathrm{L}^{\otimes n-1}\otimes\mathrm{L})\cong(\mathrm{L}^{\otimes n-1}\otimes\mathrm{L})\otimes(\mathrm{L}^{\otimes n-1})^{\otimes k}\\&\cong\mathrm{L}^{\otimes(n-1).k+n}\cong\mathrm{L}\otimes(\mathrm{L}^{\otimes n-1})^{\otimes(k+1)}\cong CL_{k+1}(\mathrm{L}^{\otimes n-1},\mathrm{L})\cong {_nCL_{k+1}}(\mathrm{L},K).\qquad\square\end{aligned}$$

In the case $n=2$ we recover application 3.1 in [5], since $HL_k(\mathrm{L},\mathrm{L})\cong {_2HL_{k-1}}(\mathrm{L},\mathrm{L})\cong {_2HL_k}(\mathrm{L},K)\cong HL_{k+1}(\mathrm{L},K)$.

Proposition 3.5. *For a Leibniz* n*-algebra* L *the following isomorphism holds*

$$_nHL_k(\mathrm{L},\mathrm{L}^{\otimes n-2})\cong HL_{k+1}(\mathrm{L}^{\otimes n-1}),\quad k\geqslant 0.$$

Proof. $_nHL_k(\mathrm{L},\mathrm{L}^{\otimes n-2})\cong HL_k(\mathrm{L}^{\otimes n-1},\mathrm{L}^{\otimes n-2}\otimes\mathrm{L})\cong HL_k(\mathrm{L}^{\otimes n-1},\mathrm{L}^{\otimes n-1})\cong HL_{k+1}(\mathrm{L}^{\otimes n-1})$. □

In the case $n=2$ we recover the well-known isomorphism $_2HL_k(\mathrm{L})\cong HL_{k+1}(\mathrm{L})$ in [1].

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