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Differential Geometry

Complete intersections with metrics of positive scalar curvature ☆

Fuquan Fang [a], Peng Shao [b]

[a] *Department of Mathematics, Capital Normal University, Beijing, PR China*
[b] *Chern Institute of Mathematics, Nankai University, Weijin Road 94, Tianjin 300071, PR China*

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Abstract

We give a complete list of complex projective complete intersections admitting Riemannian metrics of positive scalar curvature. ***To cite this article: F. Fang, P. Shao, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Intersections complètes admettant des métriques à courbure salaire positive. Nous donnons la liste des variétés complexes projectives intersections complètes, qui admettent une métrique riemannienne à courbure scalaire positive. ***Pour citer cet article : F. Fang, P. Shao, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

In their landmark works, Gromov and Lawson [4–6], as well as Schoen and Yau [9], made a series of fundamental contributions to the problem of when a manifold admits a Riemannian metric with positive scalar curvature. In particular, the Gromov–Lawson conjecture was raised, which was later solved by Stolz [10]. All these together shows that any simply connected closed n-manifold of dimension $n \geqslant 5$ admits a Riemannian metric of positive scalar curvature if it is not Spin, and a Spin manifold admits such a metric if and only if its Atiyah–Milnor invariant (an element of $KO^{-n}(pt)$) vanishes.

It is always interesting to study algebraic manifolds from a differential geometry point of view. In this Note we are concerned with the question of which complete intersections admit Riemannian metrics with positive scalar curvature. Recall that a complete intersection $V^n_{d_1,\dots,d_r} \subset \mathbb{CP}^{n+r}$ is the transversal intersection of hypersurfaces in the projective space defined by homogeneous polynomials of degrees $d_1, \dots, d_r$, respectively; here we call $\{d_1, \dots, d_r\}$ the multidegree. By the Barth–Lefschetz theorem, every complete intersection of dimension $\geqslant 2$ is simply connected. It turns out by using Stolz's result [10] that we only need to determine the Atiyah–Milnor invariants of complete intersections of

☆ Supported by NSF Grant of China #10671097 and the Capital Normal University.
E-mail addresses: fuquan_fang@yahoo.com (F. Fang), shallpion@hotmail.com (P. Shao).

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complex dimension at least 3. In complex dimension 2 we need some special treatment using Seiberg–Witten theory. Our main result is as follows:

Theorem 1. $V^n_{d_1,\dots,d_r}$ *admits a Riemannian metric of positive scalar curvature if and only if one of the following holds*:

(1.1) $(d_1,\dots,d_r)=(1,\dots,1)$ *if* $n=1$.

(1.2) $(d_1,\dots,d_r)=(2),(3),(2,2)$ *or* $(1,\dots,1)$ *if* $n=2$.

(1.3) $n+r+1-(d_1+\dots+d_r)$ *is odd or* $n+r+1-(d_1+\dots+d_r)$ *is even but positive, if* $n=2k\geqslant 4$.

(1.4) $n=4k+3$.

(1.5) $n=4k+1\geqslant 5$ *and* $4k+r+2-\sum d_i$ *is odd or* $4k+r+2-\sum d_i$ *is even but* $\sum\binom{(4k+r\pm d_1\pm\dots\pm d_{r-1}+d_r)/2}{4k+r+1}\equiv 0 \bmod 2$, *where the sum is taken over the* 2^{r-1} *possible terms.*

The $\hat{A}$-genus of complete intersections were calculated by Brooks and (1.3) was proved in [2]. (1.1) is trivial and (1.4) follows directly by Gromov–Lawson–Stolz [10]. For $r=1$ (1.5) was proved by Zhang [13] and for $r=2$ it was due to Feng–Zhang [3].

2. Proof of assertion (1.2)

For a complete intersection surface $V^2_{d_1,\dots,d_r}$, the Kodaira dimension is as follows ([1]):

$$\kappa\big(V^2_{d_1,\dots,d_r}\big)=\begin{cases}-\infty, & \{d_i\}=(2),(3),(2,2),(1,\dots,1);\\ 0, & \{d_i\}=(4),(2,3),(2,2,2);\\ 2, & \text{otherwise.}\end{cases}\tag{1}$$

To prove assertion (1.2) we only need to show that the Kodaira dimension must be equal to $-\infty$ if it admits a metric of positive scalar curvature.

First, it is a standard result in algebraic geometry that 2-dimensional complete intersections with multidegree $(2),(3),(2,2),(1,\dots,1)$ are diffeomorphic to $\mathbb{CP}^1\times\mathbb{CP}^1$, $\mathbb{CP}^2\sharp 5\overline{\mathbb{CP}^2}$, $\mathbb{CP}^2\sharp 6\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2$ respectively. By [4,9] the connected sum of two 4-manifolds with positive scalar curvature metrics admits also such a metric. Hence all these four manifolds admit metrics of positive scalar curvature. It remains only to prove that 2-dimensional complete intersections with Kodaira dimension 0 or 2 do not admit metrics of positive scalar curvature.

It is well known that the total Chern class of complete intersection $V^n_{d_1,\dots,d_r}$ is:

$$c\big(V^n_{d_1,\dots,d_r}\big)=(1+x)^{n+r+1}\left(\prod_{i=1}^{i=r}(1+d_ix)^{-1}\right),\tag{2}$$

where x denotes the pull-back of the Kähler class from $\mathbb{CP}^{n+r}$. It is easy to verify that complete intersections with Kodaira dimension 0 all have vanishing first Chern class, namely they are all $K3$ surfaces and have $b_2^+=3$ (the dimension of self-dual harmonic 2-forms). A well-known result in Seiberg–Witten theory (cf. [8], Corollary 5.1.8) asserts that, a Riemannian 4-manifold with positive scalar curvature does not have any non-trivial monopole, and thus, all Seiberg–Witten invariants vanish if $b_2^+>1$. On the other hand, the Seiberg–Witten invariant of any Kähler surface with $b_2^+>1$ is not trivial with the standard Spin^c-structure (cf. [11]). Therefore we need only to consider the complete intersections of general type, i.e., with Kodaira dimension 2, and with $b_2^+\leqslant 1$. Obviously, every complete intersection surface satisfies $b_2^+\geqslant 1$.

Let S denote a general type surface S with $b_2^+(S)=1$. Recall the Miyaoka–Yau inequality $c_1^2(S)\leqslant 3c_2(S)$ (cf. [7]). By the signature theorem $\frac{1}{3}p_1(S)=\sigma(S)=\frac{1}{3}(c_1^2(S)-2c_2(S))$. Note that S is simply connected, so the first Betti number $b_1(S)=0$. This together with the Miyaoka–Yau inequality shows

$$\frac{4}{3}c_1^2(S)\leqslant c_1^2(S)+c_2(S)=3\big(\sigma(S)+\chi(S)\big)=6\big(1-b_1(S)+b_2^+(S)\big)=12,\tag{3}$$

where $\sigma(S)$, $\chi(S)$ denote the signature and Euler number respectively. On the other hand, if $S = V^2_{d_1,\dots,d_r}$, by formula (2) we have

$$c_1^2\big(V^2_{d_1,\dots,d_r}\big) = \left(\sum_1^r d_i - (r+3)\right)^2 \prod d_i$$

and by (3)

$$\left(\sum_1^r (d_i - 1) - 3\right)^2 \prod d_i \leqslant 9.$$

This restricts our attention to very few possible surfaces of general type, and a careful check by (2) shows that none of them can have $b_2^+ = 1$.

3. Proof of assertion (1.5)

According to Zhang [12], we say that a pair of manifolds (K, B) is a *characteristic pair* if they satisfy the following conditions:

- $\dim(K) = 8k+4$, $\dim(B) = 8k+2$.
- K is an oriented Spin^c manifold with a Spin^c-structure $c \in H^2(K;\mathbb{Z})$.
- B is a submanifold of K and $[B] \in H_{8k+2}(K,\mathbb{Z})$ is the Poincaré dual of c.

It is easy to see that B is a spin manifold. Let $\hat{\mathscr{A}}(B) \in KO^{-2}(pt) \cong \mathbb{Z}_2$ denote the Atiyah–Milnor invariant of B. In [12] Zhang found the following remarkable formula

$$\hat{\mathscr{A}}(B) \equiv \big\langle \hat{A}(K)\exp(c/2), [K]\big\rangle \mod 2. \tag{4}$$

By formula (2) we know that $V^{4k+1}_{d_1,\dots,d_r}$ is a Spin manifold if and only if $4k+r+2-\sum d_i$ is even. Now we are going to use Zhang's formula (4) to calculate the Atiyah–Milnor invariant of a spin complete intersection $V^{4k+1}_{d_1,\dots,d_r}$.

Observe that $V^{4k+1}_{d_1,\dots,d_r} \subset V^{4k+2}_{d_1,\dots,d_{r-1}}$ is a submanifold of codimension 2, Poincaré dual to $d_r x$. With $c = d_r x$, it is easy to see that $(V^{4k+2}_{d_1,\dots,d_{r-1}}, V^{4k+1}_{d_1,\dots,d_r})$ is a characteristic pair in the sense of Zhang [12] mentioned above. Therefore, the formula (4) implies

$$\hat{\mathscr{A}}\big(V^{4k+1}_{d_1,\dots,d_r}\big) \equiv \left\langle \hat{A}\big(V^{4k+2}_{d_1,\dots,d_{r-1}}\big)\exp\left(\frac{d_r x}{2}\right), \big[V^{4k+2}_{d_1,\dots,d_{r-1}}\big]\right\rangle \mod 2. \tag{5}$$

Note that the normal bundle of $V^{4k+2}_{d_1,\dots,d_{r-1}} \subset \mathbb{CP}^{4k+1+r}$ is $H^{d_1} \oplus \cdots \oplus H^{d_{r-1}}$, where $H^{d_1}, \dots, H^{d_{r-1}}$ are the complex line bundles with first Chern classes $d_1 x, \dots, d_{r-1}x$ respectively. Therefore, the stable tangent bundle of $V^{4k+2}_{d_1,\dots,d_{r-1}}$ is isomorphic to $(4k+r+2)H - (H^{d_1} + \cdots + H^{d_{r-1}})$. Note that $V^{4k+2}_{d_1,\dots,d_{r-1}} \subset \mathbb{CP}^{4k+1+r}$ is a submanifold representing the Poincaré dual of $d_1 \cdots d_{r-1} x^{r-1}$. It is easy to check that the right hand side of Eq. (5) equals (all calculations are in $\mathbb{Z}/2$)

$$\begin{aligned}
\hat{\mathscr{A}}\big(V^{4k+1}_{d_1,\dots,d_r}\big) &= 2^{r-1}\left\langle \left(\frac{\frac{x}{2}}{\sinh\frac{x}{2}}\right)^{4k+r+2} \sinh\left(\frac{d_1 x}{2}\right)\cdots\sinh\left(\frac{d_{r-1}x}{2}\right)\exp\left(\frac{d_r x}{2}\right), \big[\mathbb{CP}^{4k+1+r}\big]\right\rangle \\
&= \frac{1}{2^{4k+2}}\left\langle \left(\frac{x}{\sinh x}\right)^{4k+r+2} \sinh(d_1 x)\cdots\sinh(d_{r-1}x)\exp(d_r x), \big[\mathbb{CP}^{4k+1+r}\big]\right\rangle.
\end{aligned} \tag{6}$$

We use residue integral to calculate this integral, since what we are interested in is nothing but some coefficient in a polynomial. We use $\Gamma(0)$ (resp. $\Gamma(1)$) to denote a small circle around 0 (resp. 1).

$$\begin{aligned}
\hat{\mathcal{A}}\big(V^{4k+1}_{d_1,\dots,d_r}\big) &= \frac{2}{2\pi\sqrt{-1}} \oint\limits_{\Gamma(0)} \frac{e^{d_r z}}{(e^z - e^{-z})^{4k+r+2}} \prod_{i=1}^{r-1}\big(e^{d_i z} - e^{-d_i z}\big)\,\mathrm{d}z \\
&= \frac{2}{2\pi\sqrt{-1}} \oint\limits_{\Gamma(0)} \frac{e^{(4k+r+1+d_r)z}}{(e^{2z} - 1)^{4k+r+2}} \prod_{i=1}^{r-1}\big(e^{d_i z} - e^{-d_i z}\big)\,\mathrm{d}e^z \\
&= \frac{2}{2\pi\sqrt{-1}} \oint\limits_{\Gamma(1)} \frac{t^{4k+r+1+d_r}\prod_{i=1}^{r-1}(t^{d_i} - t^{-d_i})}{(t^2 - 1)^{4k+r+2}}\,\mathrm{d}t \\
&= \frac{1}{2\pi\sqrt{-1}} \oint\limits_{\Gamma(1)} \frac{\sum t^{4k+r\pm d_1\pm\cdots\pm d_{r-1}+d_r}}{(t^2 - 1)^{4k+r+2}}\,\mathrm{d}t^2.
\end{aligned} \tag{7}$$

Since $4k + r + 2 - \sum d_i$ is an even number, so there is no risk in writing this integral as:

$$\begin{aligned}
\hat{\mathcal{A}}\big(V^{4k+1}_{d_1,\dots,d_r}\big) &= \frac{1}{2\pi\sqrt{-1}} \oint\limits_{\Gamma(0)} \frac{\sum (w+1)^{(4k+r\pm d_1\pm\cdots\pm d_{r-1}+d_r)/2}}{w^{4k+r+2}}\,\mathrm{d}w \\
&= \sum \binom{(4k + r \pm d_1 \pm \cdots \pm d_{r-1} + d_r)/2}{4k + r + 1},
\end{aligned} \tag{8}$$

here $\binom{n}{m} = \frac{n\cdots(n-m+1)}{m!}$ and we sum over all the possibilities $\pm d_1 \pm \cdots \pm d_{r-1} + d_r$. Then by [10] the proof of assertion (1.5) is complete.

Remark 2. One should note that the choice of d_r is not important to the result because, for positive integer n, $\binom{\frac{n-1}{2}+\alpha}{n} \equiv \frac{(\frac{n-1}{2}+\alpha)(\frac{n-1}{2}+\alpha-1)\cdots(\frac{n-1}{2}+\alpha-n+1)}{n!} \equiv \frac{(\frac{n-1}{2}-\alpha)\cdots(\frac{n-1}{2}-\alpha-n+1)}{n!} \equiv \binom{\frac{n-1}{2}-\alpha}{n} \bmod 2$.

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References

[1] W. Barth, C. Peters, A. Van de Ven, Compact Complex Surfaces, Springer-Verlag, Berlin, 1984.
[2] R. Brooks, The $\hat{A}$-genus of complex hypersurfaces and complete intersections, Proc. Amer. Math. Soc. 87 (1983) 528–532.
[3] H. Feng, B. Zhang, Existence of Riemannian metrics with positive scalar curvature of complex complete intersections, Adv. Math. (China) 36 (2007) 47–50.
[4] M. Gromov, B. Lawson, Spin and scalar curvature in the presence of a fundamental group. I, Ann. Math. 111 (1980) 209–230.
[5] M. Gromov, B. Lawson, The classification of simply connected manifolds of positive scalar curvature, Ann. Math. 111 (1980) 423–434.
[6] M. Gromov, B. Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Inst. Hautes Études Sci. Publ. Math. 58 (1983) 83–196.
[7] Y. Miyaoka, On the Chern numbers of surfaces of general type, Invent. Math. 42 (1977) 225–237.
[8] J. Morgan, The Seiberg–Witten Equations and Applications to the Topology of Smooth Four-Manifolds, Princeton Press, 1996.
[9] R. Schoen, S.T. Yau, The structure of manifolds with positive scalar curvature, Manuscripta Math. 28 (1979) 159–183.
[10] S. Stolz, Simply connected manifolds of positive scalar curvature, Ann. Math. 136 (1992) 511–540.
[11] C. Taubes, The Seiberg–Witten invariants and symplectic forms, Math. Res. Lett. 1 (6) (1994) 809–822.
[12] W. Zhang, Spin^c-manifolds and Rokhlin congruences, C. R. Acad. Sci. Paris, Sér. I Math. 317 (1993) 689–692.
[13] W. Zhang, Existence of Riemannian metrics with positive scalar curvature on complex hypersurfaces, Acta Math. Sinica (Chin. Ser.) 39 (4) (1996) 460–462.