



Mathematical Problems in Mechanics

The pure displacement problem in nonlinear three-dimensional elasticity: intrinsic formulation and existence theorems

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Abstract

In this Note, the equations of nonlinear three-dimensional elasticity corresponding to the pure displacement problem are recast either as a boundary value problem, or as a minimization problem, where the unknown is in both cases the Cauchy–Green strain tensor, instead of the deformation as is customary. We then show that either problem possesses a solution if the applied forces are sufficiently small and the stored energy function satisfies specific hypotheses. The second problem provides an example of a minimization problem for a non-coercive functional over a Banach manifold. **To cite this article:** P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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Résumé

Le problème en déplacement pur en élasticité non linéaire tri-dimensionnelle : Formulation intrinsèque et théorèmes d'existence. Dans cette Note, les équations de l'élasticité non linéaire tri-dimensionnelle correspondant au problème en déplacement pur sont ré-écrites, soit comme un problème aux limites, soit comme un problème de minimisation, l'inconnue étant dans les deux cas le tenseur des déformations de Cauchy–Green, au lieu de la déformation comme il est usuel. On montre ensuite que l'un et l'autre problème ont au moins une solution si les forces sont suffisamment petites et si la densité d'énergie satisfait certaines hypothèses naturelles. Le second problème constitue un exemple de problème de minimisation d'une fonctionnelle non coercive sur une variété de Banach. **Pour citer cet article :** P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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On trouvera les démonstrations complètes des résultats annoncés ici dans [10]. On renvoie par ailleurs à la version anglaise pour les numéros de formules et les références non citées ici. Dans ce qui suit, Ω désigne un ouvert borné et simplement connexe de \mathbb{R}^3 , de frontière $\partial\Omega$ suffisamment régulière. Le problème en déplacement pur de l'élasticité tri-dimensionnelle se formule classiquement sous la forme d'un problème aux limites, comprenant les équations d'équilibre et la condition aux limites (1), et la loi de comportement (3), exprimée en fonction du *tenseur des déforma-*

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tions de Cauchy–Green $\mathbf{C}(x)$ défini en tout point x de la configuration de référence $\overline{\Omega}$ par la formule (2). L’inconnue principale est alors le *champ de déformations* $\varphi : \overline{\Omega} \rightarrow \mathbb{R}^3$.

Dans cette Note, on se propose de reformuler le problème (1)–(3), ainsi que le problème de minimisation associé lorsque le matériau considéré est hyperélastique, en prenant comme inconnue principale le *champ de tenseurs* $\mathbf{C} : \overline{\Omega} \rightarrow \mathbb{S}_>^3$; on obtient ainsi une *formulation intrinsèque* de chacun de ces deux problèmes (l’idée de telles formulations remonte à Antman [2]). A cette fin, on doit commencer par définir les *champs de tenseurs admissibles*. Comme le montre la relation (11), ces champs appartiennent à l’espace de Sobolev $W^{2,s}(\Omega; \mathbb{S}_>^3)$, où $s > 3$, ils annulent (au sens des distributions) le tenseur de Riemann de composantes $R_{ijk}^p(\mathbf{C})$, et enfin ils vérifient des relations appropriées le long de la surface $S = \partial\Omega$, qui expriment que les formes fondamentales des surfaces $\mathbf{id}(\partial\Omega)$ et $\varphi(\partial\Omega)$ coïncident (c’est ainsi qu’on prend en compte la condition aux limites $\varphi = \mathbf{id}$ sur $\partial\Omega$; un usage essentiel est fait ici de l’extension au sens des distributions du théorème fondamental de la théorie des surfaces due à S. Mardare [14]). En procédant comme dans C. Mardare [12], on montre ensuite que l’ensemble $\mathbb{T}(\Omega)$ formé par ces champs de tenseurs admissibles est une *variété de Banach* (Théorème 2), et que la formulation intrinsèque du problème en déplacement pur (1)–(3) est constituée par les relations (11)–(12), l’application \mathcal{G} étant l’inverse de l’application \mathcal{F} définie en (8). On montre enfin (Théorème 4) que cette formulation intrinsèque a une solution si les forces sont suffisamment petites dans l’espace $W^{1,s}(\Omega; \mathbb{R}^3)$. La démonstration utilise en particulier le *théorème des fonctions implicites dans une variété de Banach* de Abraham, Marsden et Ratiu [1].

On suppose ensuite que le matériau est hyperélastique, avec une densité d’énergie de la forme (15), proposée par Ciarlet et Geymonat [8]. On établit alors que la formulation intrinsèque du problème de minimisation associé, qui consiste à minimiser la fonctionnelle \mathcal{I} définie en (18) sur la variété $\mathbb{T}(\Omega)$ de (11), a lui aussi une solution si les forces sont à nouveau suffisamment petites dans l’espace $W^{1,s}(\Omega; \mathbb{R}^3)$ (Théorème 5). La démonstration repose entre autres sur le théorème fondamental d’existence de Ball [3] et sur la comparaison due à Zhang [16] entre la solution fournie par celui-ci et celle fournie par le théorème des fonctions implicites. Comme il sera montré dans l’article développé [10], le Théorème 5 s’étend aux densités d’énergie plus générales considérées par Ball [3] et Ball et Murat [4].

Il est à noter que le Théorème 5 fournit un *exemple de problème de minimisation d’une fonctionnelle non coercive sur une variété de Banach*.

1. The classical formulation of the pure displacement problem of nonlinear elasticity

All matrices, function spaces, etc., considered in this Note are real.

The notations \mathbb{M}^3 , \mathbb{M}_+^3 , \mathbb{S}^3 , and $\mathbb{S}_>^3$ respectively designate the space of all square matrices of order three, the set of all matrices $\mathbf{F} \in \mathbb{M}^3$ with $\det \mathbf{F} > 0$, the space of all symmetric matrices of order three, and the set of all positive definite symmetric matrices of order three.

Consider an elastic body, which in the absence of applied forces occupies the closure of a bounded and connected open subset Ω of \mathbb{R}^3 , called the *reference configuration* of the body. A *deformation* of the elastic body is a smooth enough mapping $\varphi : \overline{\Omega} \rightarrow \mathbb{R}^3$ that is *orientation preserving* (i.e., $\det \nabla \varphi(x) > 0$ for all $x \in \Omega$) and *injective* on the open set Ω (i.e., no interpenetration of matter occurs). We consider here the *pure displacement problem*, i.e., we assume that every *admissible deformation* φ satisfies $\varphi(x) = x$ for all $x \in \partial\Omega$, or in short $\varphi = \mathbf{id}$ on $\partial\Omega$.

We assume that the body is subjected to *applied body forces of dead load type*, given by their densities $\mathbf{f} : \Omega \rightarrow \mathbb{R}^3$ per unit volume. Thanks to the *stress principle of Euler and Cauchy* and to *Cauchy’s theorem*, there exists a (second Piola–Kirchhoff) *stress tensor field* $\boldsymbol{\Sigma} : \overline{\Omega} \rightarrow \mathbb{S}^3$ that satisfies the following *equations of equilibrium in the reference configuration*:

$$-\operatorname{div}(\nabla \varphi(x) \boldsymbol{\Sigma}(x)) = \mathbf{f}(x), \quad x \in \Omega, \quad \text{and} \quad \varphi(x) = x, \quad x \in \partial\Omega. \quad (1)$$

The above equations of equilibrium must be supplemented by the *constitutive equation* of the elastic material, relating the stress tensor field $\boldsymbol{\Sigma}$ and the deformation φ by means of a function $\hat{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{M}_+^3 \rightarrow \mathbb{S}^3$, called the *response function* of the material, as $\boldsymbol{\Sigma}(x) = \hat{\boldsymbol{\Sigma}}(x, \nabla \varphi(x))$ for all $x \in \overline{\Omega}$.

Because of the *principle of material frame-indifference*, the stress tensor $\boldsymbol{\Sigma}(x)$ at any point x of the reference configuration $\overline{\Omega}$ depends on the deformation gradient $\nabla \varphi(x)$ only via its associated *Cauchy–Green tensor*

$$\mathbf{C}(x) := (\nabla \varphi(x))^T \nabla \varphi(x). \quad (2)$$

In other words, there exists a *response function* $\tilde{\Sigma} : \bar{\Omega} \times \mathbb{S}_>^3 \rightarrow \mathbb{S}^3$ such that

$$\Sigma(x) = \tilde{\Sigma}(x, \mathbf{C}(x)), \quad x \in \bar{\Omega}. \tag{3}$$

The system formed by Eqs. (1)–(3) constitute the *equations of nonlinear three-dimensional elasticity for the pure displacement problem* (for more details about the derivation of these equations, see, e.g., [5]).

In the *classical approach*, the tensor $\Sigma(x)$ appearing in the equations of equilibrium (1) is replaced with its expression (3), so that the deformation $\varphi : \bar{\Omega} \rightarrow \mathbb{R}^3$ becomes the *primary unknown*.

2. The manifold of admissible Cauchy–Green tensor fields; intrinsic formulation of the pure displacement problem

Our aim is to recast the pure displacement problem (1)–(3) (and later on, its formulation as a minimization problem; cf. Section 4) *in terms of the matrix field* $\mathbf{C} : \bar{\Omega} \rightarrow \mathbb{S}_>^3$ *as the primary unknown*, instead of the deformation φ as in the classical approach; this is the basis of the so-called *intrinsic approach*, first suggested, albeit briefly, by Antman [2].

Complete proofs and various generalizations will be found in [10]. See also [11] for the “linearized version” of the results announced here, which serves as a useful complement to [7] where only the pure traction problem was considered.

To begin with, we need to characterize those matrix fields that are Cauchy–Green tensor fields induced by those deformations that are admissible for the pure displacement problem in nonlinear elasticity.

In all that follows, we assume that Ω is a *bounded and simply-connected open* subset of \mathbb{R}^3 with a *sufficiently smooth boundary*. Latin indices and exponents take their values in the set $\{1, 2, 3\}$. Recall that the Sobolev space $W^{m,s}(\Omega)$ is an algebra if $ms > 3$ since Ω is a three-dimensional domain.

The set of *admissible deformations* that is best suited for our subsequent purposes turns out to be defined by

$$\mathbf{D}(\Omega) := \{ \varphi \in W^{3,s}(\Omega; \mathbb{R}^3); \det \nabla \varphi(x) > 0 \text{ for all } x \in \bar{\Omega}, \varphi(x) = x, x \in \partial\Omega \}, \tag{4}$$

for some $s > 3/2$. With any deformation $\varphi \in \mathbf{D}(\Omega)$, we associate the *Christoffel symbols* Γ_{ij}^k and the mixed components R_{ijk}^p of the *Riemann tensor field* by letting

$$g_{ij} := (\mathbf{C})_{ij}, \quad (g^{k\ell}) := (g_{ij})^{-1}, \quad \Gamma_{ij}^k := \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}), \tag{5}$$

$$R_{ijk}^p(\mathbf{C}) := \partial_j \Gamma_{ik}^p - \partial_k \Gamma_{ij}^p + \Gamma_{ik}^\ell \Gamma_{j\ell}^p - \Gamma_{ij}^\ell \Gamma_{k\ell}^p, \tag{6}$$

where $\mathbf{C} := \nabla \varphi^T \nabla \varphi$. The corresponding *set of admissible Cauchy–Green tensor fields* is then naturally defined as the image

$$\mathbb{T}(\Omega) := \mathcal{F}(\mathbf{D}(\Omega)) \tag{7}$$

through the mapping

$$\mathcal{F} : \varphi \in W^{3,s}(\Omega; \mathbb{R}^3) \mapsto \mathcal{F}(\varphi) := \nabla \varphi^T \nabla \varphi \in W^{2,s}(\Omega; \mathbb{S}^3). \tag{8}$$

It is clear that each matrix field $\mathbf{C} \in \mathbb{T}(\Omega)$ is continuous over $\bar{\Omega}$ (in the sense that the equivalence class \mathbf{C} contains one and only one matrix field that is continuous over $\bar{\Omega}$) and that the matrix $\mathbf{C}(x)$ is positive definite at all $x \in \bar{\Omega}$. In addition, it is well known that the components g_{ij} of the matrix field \mathbf{C} necessarily satisfies the equations

$$R_{ijk}^p(\mathbf{C}) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

It remains to recast the boundary condition $\varphi = \mathbf{id}$ on $\partial\Omega$ in terms of the matrix field \mathbf{C} .

The *fundamental theorem of surface theory* asserts that a surface is uniquely determined up to a rigid motion in \mathbb{R}^3 by its two fundamental forms. The condition $\varphi = \mathbf{id}$ on $\partial\Omega$ is thus equivalent, up to a rigid motion in \mathbb{R}^3 , to the condition that the two fundamental forms defined by the restriction $\varphi|_{\partial\Omega}$ to $\partial\Omega$ of the immersion φ (note that $\varphi \in \mathbf{D}(\Omega)$ implies that $\varphi \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R}^3)$) coincide with the two fundamental forms defined by the immersion $\mathbf{id}|_{\partial\Omega}$. Recall that the first fundamental form induced by the immersion φ at a point $x \in S$ of the surface $S := \partial\Omega$ is the restriction to the space $T_x S \times T_x S$ of the bilinear form

$$\mathbf{A}_x(\mathbf{C}) : (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbf{a}^T \mathbf{C}(x) \mathbf{b} \in \mathbb{R},$$

where $T_x S$ denotes the tangent space of S at $x \in S$, and that the second fundamental form induced by the immersion φ at a point $x \in S$ of the surface $S := \partial\Omega$ is the restriction to the space $T_x S \times T_x S$ of the bilinear form

$$\mathbf{B}_x(\mathbf{C}) : (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto -\frac{1}{2} \mathbf{a}^T (\mathcal{L}_{\mathbf{n}} \mathbf{C})(x) \mathbf{b} \in \mathbb{R},$$

where \mathcal{L} denotes the Lie derivative and \mathbf{n} is a C^1 -extension in a neighborhood of S of a vector field that is unit and normal to S with respect to the metric induced by the field \mathbf{C} (i.e., $\mathbf{a}^T \mathbf{C}(x) \mathbf{n} = 0$ for all $\mathbf{a} \in T_x S$ and $\mathbf{n}^T \mathbf{C}(x) \mathbf{n} = 1$). To fix the sign of the second fundamental form, we choose \mathbf{n} pointing inwardly to Ω . Thus the boundary condition $\varphi = \mathbf{id}$ on $\partial\Omega$ is equivalent, up to a rigid motion in \mathbb{R}^3 , to the relations

$$\mathbf{A}_x(\mathbf{C}) = \mathbf{A}_x(\mathbf{I}) \quad \text{and} \quad \mathbf{B}_x(\mathbf{C}) = \mathbf{B}_x(\mathbf{I}) \quad \text{on } T_x S \times T_x S \quad \text{for all } x \in S. \quad (9)$$

We can then prove that the above *necessary* conditions are in fact *sufficient*. To this end, we use the extensions in the sense of distributions of the fundamental theorems of Riemannian geometry and of surface theory, both due to S. Mardare (see [13–15]; this is where the assumption $s > 3/2$ is needed, so as to guarantee that the trace on S of a function in $W^{1,s}(\Omega)$ is in $L^\sigma(S)$ with $\sigma > 2$). Naturally, the assumption that Ω is simply connected is also crucially needed here. We then have:

Theorem 1. *Let $s > 3/2$. A matrix field $\mathbf{C} \in W^{2,s}(\Omega; \mathbb{S}^3)$ belongs to the image*

$$\mathbb{T}(\Omega) := \mathcal{F}(\mathbf{D}(\Omega)) \quad (10)$$

if and only if $\mathbf{C}(x) \in \mathbb{S}_{>}^3$ for all $x \in \overline{\Omega}$, $R_{ijk}^p(\mathbf{C}) = 0$ in $\mathcal{D}'(\Omega)$, and relations (9) are satisfied.

The next theorem shows that the above set $\mathbb{T}(\Omega)$ of *admissible Cauchy–Green tensor fields for the pure displacement problem of nonlinear elasticity*, as defined in (10), is a *Banach manifold*:

Theorem 2. *Let $s > 3/2$, and let the set $\mathbf{D}(\Omega)$ and the mapping \mathcal{F} be defined as in (4) and (8). Then the set $\mathbb{T}(\Omega) = \mathcal{F}(\mathbf{D}(\Omega))$ is a Banach manifold of class C^∞ in the Banach space $W^{2,s}(\Omega; \mathbb{S}^3)$, and the mapping \mathcal{F} is a C^∞ -diffeomorphism from $\mathbf{D}(\Omega)$ onto $\mathbb{T}(\Omega)$.*

Sketch of proof. It suffices to prove that the set $\mathbf{D}(\Omega)$ is itself a C^∞ -manifold in the Banach space $W^{3,s}(\Omega; \mathbb{R}^3)$, and that the mapping \mathcal{F} is an embedding of class C^∞ . To this end, we essentially proceed as in C. Mardare [12], by successively proving the following steps:

(i) *The set $\mathbf{D}(\Omega)$ is a manifold of class C^∞ in the Banach space $W^{3,s}(\Omega; \mathbb{R}^3)$ and the tangent space to $\mathbf{D}(\Omega)$ at any $\varphi \in \mathbf{D}(\Omega)$ is the space $W^{3,s}(\Omega; \mathbb{R}^3) \cap W_0^{1,s}(\Omega; \mathbb{R}^3)$. This is so because $\mathbf{D}(\Omega)$ is open in the closed affine space $\{\mathbf{id} + W^{3,s}(\Omega; \mathbb{R}^3) \cap W_0^{1,s}(\Omega; \mathbb{R}^3)\}$.*

(ii) *The mapping \mathcal{F} is a homeomorphism from $\mathbf{D}(\Omega)$ onto its image.* To prove this, we need to prove that \mathcal{F} is injective, continuous, and that its inverse $\mathcal{G} := \mathcal{F}^{-1} : \mathcal{F}(\mathbf{D}(\Omega)) \rightarrow W^{3,s}(\Omega; \mathbb{R}^3)$ is also continuous (proving the continuity of \mathcal{G} relies in particular on an argument similar to that used in Ciarlet and Mardare [9], although different function spaces were used there).

(iii) *At every $\varphi \in \mathbf{D}(\Omega)$, the tangent mapping $T_\varphi \mathcal{F} : W^{3,s}(\Omega; \mathbb{R}^3) \cap W_0^{1,s}(\Omega; \mathbb{R}^3) \rightarrow W^{2,s}(\Omega; \mathbb{S}^3)$ is injective.* Since the mapping \mathcal{F} is bilinear, it is easy to prove that its tangent map at φ is defined by $(T_\varphi \mathcal{F})(\chi) = \nabla \varphi^T \nabla \chi + \nabla \chi^T \nabla \varphi$ for all $\chi \in W^{3,s}(\Omega; \mathbb{R}^3)$. Since $\varphi = \mathbf{id}$ on $\partial\Omega$ and $\det \nabla \varphi(x) > 0$ for all $x \in \overline{\Omega}$, the mapping φ is a C^1 -diffeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}$. It then suffices to use the *three-dimensional Korn inequality in curvilinear coordinates* (cf. Ciarlet [6]; here the curvilinear coordinates are those defined by the immersion φ).

(iv) *The tangent mapping $T_\varphi \mathcal{F}$ has a closed split range in the space $W^{2,s}(\Omega; \mathbb{S}^3)$.* To prove this, we need to prove that the range A of $T_\varphi \mathcal{F}$ is a closed subset of the space $W^{2,s}(\Omega; \mathbb{S}^3)$ and that there exists a closed subspace B of the same space such that $W^{2,s}(\Omega; \mathbb{S}^3) = A \oplus B$; cf. Abraham, Marsden and Ratiu [1, Definition 2.1.14].

(v) *Conclusion.* The tangent mapping $T_\varphi \mathcal{F}$ being injective and having a closed split range at every $\varphi \in \mathbf{D}(\Omega)$, the mapping \mathcal{F} is an immersion, according to [1, Definition 3.5.6]. Since it is also a homeomorphism onto its image, \mathcal{F} is in fact an embedding; cf. [1, Definition 3.5.9]. Hence its image $\mathbb{T}(\Omega)$ is a manifold in the Banach space $W^{2,s}(\Omega; \mathbb{S}^3)$. This manifold is of class C^∞ since \mathcal{F} is of class C^∞ . That the mapping \mathcal{F} is a diffeomorphism of class C^∞ is a consequence of the inverse function theorem of [1, Theorem 3.5.1]. \square

Thanks to Theorems 1 and 2, we can now recast the pure displacement problem of nonlinear elasticity of Section 1 with \mathbf{C} as its primary unknown in the form of relations (11)–(12) below, which constitute the *intrinsic formulation of the pure displacement problem*.

Theorem 3. *Let $s > 3/2$, and let $\mathcal{G} := \mathcal{F}^{-1} : \mathbb{T}(\Omega) \rightarrow \mathbf{D}(\Omega)$ (cf. Theorem 2). Then a deformation $\boldsymbol{\varphi} \in \mathbf{D}(\Omega)$ is a solution to the pure displacement problem (1)–(3) if and only if its associated Cauchy–Green tensor field $\mathbf{C} = \nabla \boldsymbol{\varphi}^T \nabla \boldsymbol{\varphi}$ satisfies the following relations*

$$\mathbf{C} \in \mathbb{T}(\Omega) = \{ \mathbf{C} \in W^{2,s}(\Omega; \mathbb{S}_{>}^3); R_{.ijk}^p(\mathbf{C}) = 0 \text{ in } \mathcal{D}'(\Omega), \mathbf{A}_x(\mathbf{C}) = \mathbf{A}_x(\mathbf{I}) \text{ and } \mathbf{B}_x(\mathbf{C}) = \mathbf{B}_x(\mathbf{I}) \text{ on } T_x S \times T_x S \text{ for all } x \in S \}, \tag{11}$$

$$-\operatorname{div}\{\nabla \mathcal{G}(\mathbf{C}) \tilde{\boldsymbol{\Sigma}}(\cdot, \mathbf{C})\} = \mathbf{f} \text{ in } \mathcal{D}'(\Omega; \mathbb{R}^3). \tag{12}$$

3. Existence of solutions to the intrinsic formulation of the pure displacement problem

From now on, we assume that the elastic material is *homogeneous* and *isotropic*, and that the reference configuration $\bar{\Omega}$ is a *natural state*, so that the behavior of the material “for small strains” is governed by its two Lamé constants $\lambda \geq 0$ and $\mu > 0$ (cf., e.g., Ciarlet [5, Chapter 3]). We further assume that the material constituting the body is *hyperelastic*, with a stored energy function of the form proposed by Ciarlet and Geymonat [8], viz.,

$$\hat{W}(x, \mathbf{F}) := a|\mathbf{F}|^2 + b|\operatorname{Cof} \mathbf{F}|^2 + c(\det \mathbf{F})^2 - d \log(\det \mathbf{F}) - (3a + 3b + c) \text{ for all } (x, \mathbf{F}) \in \bar{\Omega} \times \mathbb{M}_+^3, \tag{13}$$

where $|\mathbf{A}|$ designates the Frobenius norm of a matrix $\mathbf{A} \in \mathbb{M}^3$, and the constants $a > 0, b > 0, c > 0$ and $d > 0$ are so chosen that

$$\hat{W}(x, \mathbf{F}) = \frac{\lambda}{2}(\operatorname{tr} \mathbf{E})^2 + \mu \operatorname{tr}(\mathbf{E}^2) + o(|\mathbf{E}|^2) \text{ with } \mathbf{E} := \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) \text{ for all } (x, \mathbf{F}) \in \bar{\Omega} \times \mathbb{M}_+^3. \tag{14}$$

Note that the function \hat{W} is independent of $x \in \bar{\Omega}$ and depends on $\mathbf{F} \in \mathbb{M}_+^3$ only via $\mathbf{C} := \mathbf{F}^T \mathbf{F}$. Indeed, a simple computation shows that $\hat{W}(x, \mathbf{F}) = \tilde{W}(\mathbf{C})$ for all $(x, \mathbf{F}) \in \bar{\Omega} \times \mathbb{M}_+^3$, where

$$\tilde{W}(\mathbf{C}) := a \operatorname{tr} \mathbf{C} + b \operatorname{tr} \operatorname{Cof} \mathbf{C} + c \det \mathbf{C} - \frac{d}{2} \log \det \mathbf{C} - (3a + 3b + c) \text{ for all } \mathbf{C} \in \mathbb{S}_{>}^3. \tag{15}$$

Note that the stored energy of (13) is chosen here essentially for the sake of brevity; otherwise more general hyperelastic materials, such as those considered in Ball [3] and Ball and Murat [4], can be as well considered; cf. [10].

We now show that the intrinsic formulation of the pure displacement problem has a solution provided that the body force density is “small enough” in *ad hoc* norm.

Theorem 4. *Let $s > 3/2$, and let the response function $\tilde{\boldsymbol{\Sigma}} : \bar{\Omega} \times \mathbb{S}_{>}^3 \rightarrow \mathbb{S}^3$ appearing in the constitutive equation (3) be given by*

$$\tilde{\boldsymbol{\Sigma}}(x, \mathbf{C}) = \frac{1}{2} \frac{\partial \tilde{W}}{\partial \mathbf{C}}(\mathbf{C}) \text{ for all } (x, \mathbf{C}) \in \bar{\Omega} \times \mathbb{S}_{>}^3, \tag{16}$$

where the function $\tilde{W} : \mathbb{S}_{>}^3 \rightarrow \mathbb{R}$ is given by (15).

Then there exist two constants $\varepsilon > 0$ and $\delta > 0$ such that, for each $\mathbf{f} \in W^{1,s}(\Omega; \mathbb{R}^3)$ with $\|\mathbf{f}\|_{W^{1,s}(\Omega; \mathbb{R}^3)} < \varepsilon$, there exists a unique solution $\mathbf{C} \in \mathbb{T}(\Omega)$ to the intrinsic formulation of the pure displacement problem (11)–(12) that satisfies $\|\mathbf{C} - \mathbf{I}\|_{W^{2,s}(\Omega; \mathbb{S}^3)} < \delta$.

Idea of the proof. The idea is to apply the *implicit function theorem on Banach manifolds* (cf. Abraham, Marsden and Ratiu [1, Theorem 3.5.1]) to the mapping

$$\mathcal{H} : \mathbf{C} \in \mathbb{T}(\Omega) \mapsto \mathcal{H}(\mathbf{C}) := -\operatorname{div}\{\nabla \mathcal{G}(\mathbf{C}) \tilde{\boldsymbol{\Sigma}}(\cdot, \mathbf{C})\} \in W^{1,s}(\Omega; \mathbb{R}^3),$$

in a neighborhood of $\mathbf{I} \in \mathbb{T}(\Omega)$. To this end, we need to prove that \mathcal{H} is at least of class C^1 and that its tangent mapping at \mathbf{I} , which is given by

$$T_{\mathbf{I}} \mathcal{H} : \mathbf{D} \in T_{\mathbf{I}}(\mathbb{T}(\Omega)) = (T_{\operatorname{Id}} \mathcal{F})(W^{3,s}(\Omega; \mathbb{R}^3) \cap W_0^{1,s}(\Omega; \mathbb{R}^3)) \mapsto -\operatorname{div}\left\{\frac{\partial \tilde{\boldsymbol{\Sigma}}}{\partial \mathbf{C}}(\cdot, \mathbf{I}) \mathbf{D}\right\} \in W^{1,s}(\Omega; \mathbb{R}^3),$$

is an isomorphism. To see this, we use that $\mathcal{H} = \mathcal{K} \circ \mathcal{G}$, where the mapping \mathcal{K} is defined by

$$\mathcal{K} : \boldsymbol{\varphi} \in \mathbf{D}(\Omega) \mapsto \mathcal{K}(\boldsymbol{\varphi}) := -\operatorname{div}\{\nabla \boldsymbol{\varphi} \tilde{\boldsymbol{\Sigma}}(\cdot, \nabla \boldsymbol{\varphi}^T \nabla \boldsymbol{\varphi})\} \in W^{1,s}(\Omega; \mathbb{R}^3).$$

As shown in Ciarlet [5, Theorem 4.2-2], it follows from (14) that

$$\tilde{\boldsymbol{\Sigma}}(x, \mathbf{C}) = \lambda(\operatorname{tr} \mathbf{E})\mathbf{I} + 2\mu \mathbf{E} + o(\mathbf{E}) \quad \text{for all } \mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{I}), \mathbf{C} \in \mathbb{S}_{>}^3,$$

which in turn implies that

$$\int_{\Omega} \frac{\partial \tilde{\boldsymbol{\Sigma}}}{\partial \mathbf{C}}(x, \mathbf{I}) \mathbf{e}(x) : \mathbf{e}(x) \, dx = \int_{\Omega} \left\{ \frac{\lambda}{2} (\operatorname{tr} \mathbf{e}(x))^2 + \mu |\mathbf{e}(x)|^2 \right\} \, dx \geq \mu (\|\mathbf{e}\|_{L^2(\Omega; \mathbb{S}^3)})^2$$

for all $\mathbf{e} \in L^2(\Omega; \mathbb{S}^3)$, thus a fortiori for all $\mathbf{e} \in W^{2,s}(\Omega; \mathbb{S}^3)$. This implies that the derivative of the mapping \mathcal{K} at $\boldsymbol{\varphi} = \mathbf{id}$ is an isomorphism and thus that the implicit function theorem can be applied. \square

4. Existence of solutions to the intrinsic formulation of the associated minimization problem

For a hyperelastic material, such as the one that is considered here, Eqs. (1)–(3) *formally* constitute the Euler equations associated with the critical points of the functional J defined by

$$J(\boldsymbol{\varphi}) := \int_{\Omega} \tilde{W}(\nabla \boldsymbol{\varphi}(x)^T \nabla \boldsymbol{\varphi}(x)) \, dx - \int_{\Omega} \mathbf{f}(x) \cdot \boldsymbol{\varphi}(x) \, dx, \tag{17}$$

over an appropriate set of *admissible deformations* $\boldsymbol{\varphi}$. As usual, we are interested in those critical points that *minimize* the functional J defined in (17).

The *intrinsic formulation of this minimization problem* again consists in considering that the tensor field $\mathbf{C} = \nabla \boldsymbol{\varphi}^T \nabla \boldsymbol{\varphi}$ is the primary unknown, instead of the deformation $\boldsymbol{\varphi}$. We then show that this new minimization problem, i.e., which consists in *minimizing the non-coercive functional* \mathcal{I} of (18) *over the Banach manifold* $\mathbb{T}(\Omega)$ of (11), has a solution.

Theorem 5. *Let $s > 3/2$, and let the response function $\tilde{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{S}_{>}^3 \rightarrow \mathbb{S}^3$ be given by (16), where the function $\tilde{W} : \mathbb{S}_{>}^3 \rightarrow \mathbb{R}$ is given by (15).*

Then there exist two constants $\varepsilon > 0$ and $\delta > 0$ with the following property: Given any $\mathbf{f} \in W^{1,s}(\Omega; \mathbb{R}^3)$ with $\|\mathbf{f}\|_{W^{1,s}(\Omega; \mathbb{R}^3)} < \varepsilon$, there exists a unique matrix field $\mathbf{C}_0 \in \mathbb{T}(\Omega)$ with $\|\mathbf{C}_0 - \mathbf{I}\|_{W^{2,s}(\Omega; \mathbb{S}^3)} < \delta$ that minimizes the functional \mathcal{I} defined by (recall that $\mathcal{G} = \mathcal{F}^{-1}$; cf. Theorem 3)

$$\mathcal{I}(\mathbf{C}) := \int_{\Omega} \tilde{W}(\mathbf{C}) \, dx - \int_{\Omega} \mathbf{f} \cdot \mathcal{G}(\mathbf{C}) \, dx \quad \text{for all } \mathbf{C} \in \mathbb{T}(\Omega), \tag{18}$$

over the Banach manifold $\mathbb{T}(\Omega)$ *defined in (11).*

Sketch of proof. Since all the assumptions of Theorem 4 are satisfied, there exist two constants $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, if $\|\mathbf{f}\|_{W^{1,s}(\Omega; \mathbb{R}^3)} < \varepsilon_0$, there exists a unique solution $\mathbf{C}_0 \in \mathbb{T}(\Omega)$ to problem (11)–(12) that satisfies $\|\mathbf{C}_0 - \mathbf{I}\|_{W^{2,s}(\Omega; \mathbb{S}^3)} < \delta_0$. Let

$$\mathbf{D}_0(\Omega) := \{ \boldsymbol{\varphi} \in H^1(\Omega; \mathbb{R}^3); \operatorname{Cof} \nabla \boldsymbol{\varphi} \in L^2(\Omega; \mathbb{M}^3), \det \nabla \boldsymbol{\varphi} \in L^2(\Omega), \det \nabla \boldsymbol{\varphi} > 0 \text{ a.e. in } \Omega, \boldsymbol{\varphi} = \mathbf{id} \text{ on } \partial \Omega \}.$$

On the one hand, since the function \hat{W} is polyconvex and satisfies all the assumptions of the fundamental existence theorem of Ball [3], there exists a vector field $\boldsymbol{\varphi}_0 \in \mathbf{D}_0(\Omega)$ that minimizes the functional J defined by (17) over the set $\mathbf{D}_0(\Omega)$. On the other hand, since the response function $\hat{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{M}_{+}^3 \mapsto \mathbb{S}^3$ defined by

$$\hat{\boldsymbol{\Sigma}}(x, \mathbf{F}) := \mathbf{F}^{-1} \frac{\partial \hat{W}}{\partial \mathbf{F}}(x, \mathbf{F}) \quad \text{for all } (x, \mathbf{F}) \in \overline{\Omega} \times \mathbb{M}_{+}^3,$$

satisfies the assumptions of the *implicit function theorem* as revisited by Zhang [16, Theorem 2.6], there exist two constants $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, if $\|f\|_{W^{1,s}(\Omega; \mathbb{R}^3)} < \varepsilon_1$, the boundary value problem

$$-\operatorname{div}\{\nabla\varphi \tilde{\Sigma}(\cdot, \nabla\varphi)\} = f \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^3) \quad \text{and} \quad \varphi = \mathbf{id} \quad \text{on } \partial\Omega,$$

has a unique solution $\varphi_1 \in W^{3,s}(\Omega; \mathbb{R}^3)$ satisfying $\|\varphi_1 - \mathbf{id}\|_{W^{3,s}(\Omega; \mathbb{R}^3)} < \delta_1$.

Thanks to [16, Theorem 3.4], there exists $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1)$ such that $\varphi_0 = \varphi_1$ for all $f \in W^{1,s}(\Omega; \mathbb{R}^3)$ satisfying $\|f\|_{W^{1,s}(\Omega; \mathbb{R}^3)} < \varepsilon$. Thus the vector field φ_0 satisfies relations (1), from which it follows that the matrix field $\nabla\varphi_0^T \nabla\varphi_0$ is a solution to problem (11)–(12). In addition, $\|\nabla\varphi_0^T \nabla\varphi_0 - \mathbf{I}\|_{W^{2,s}(\Omega; \mathbb{S}^3)} < \delta_0$ if ε is chosen sufficiently small. Then the uniqueness of the solution to problem (11)–(12) shows that $\nabla\varphi_0^T \nabla\varphi_0 = C_0$.

Given any matrix field $C \in \mathbb{T}(\Omega)$, there exists, thanks to Theorem 1, a vector field $\varphi \in D(\Omega)$ such that $C = \nabla\varphi^T \nabla\varphi$. Since then $\varphi \in D_0(\Omega)$, we have $J(\varphi_0) \leq J(\varphi)$. Therefore,

$$\mathcal{I}(C_0) = \mathcal{I}(\nabla\varphi_0^T \nabla\varphi_0) = J(\varphi_0) \leq J(\varphi) = \mathcal{I}(\nabla\varphi^T \nabla\varphi) = \mathcal{I}(C).$$

This shows that the tensor field C_0 is a minimizer of the functional \mathcal{I} of (18) over the set $\mathbb{T}(\Omega)$. \square

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