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Partial Differential Equations

On the regularity of the solutions to the 3D Navier–Stokes equations: a remark on the role of the helicity

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Abstract

We show that if velocity and vorticity are orthogonal at each point (and they become orthogonal fast enough) then solutions of the 3D Navier–Stokes equations are smooth. This condition implies that the helicity is identically zero and, in a certain sense, the flow resembles the 2D geometric situation. ***To cite this article: L.C. Berselli, D. Córdoba, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur la régularité des solutions des équations de Navier–Stokes 3D : une remarque sur le rôle de l'élicité. Nous démontrons que si la vélocité et le rotationnel sont perpendiculaires partout (avec une borne sur la vitesse avec laquelle ils deviennent perpendiculaires) alors les solutions des équations de Navier–Stokes 3D sont régulières. Cette condition implique que l'élicité est nulle et, dans un certain sens, que le flux ressemble à la situation géométrique bidimensionnelle. ***Pour citer cet article : L.C. Berselli, D. Córdoba, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

On considère le problème de l'existence globale de solutions régulières des équations de Navier–Stokes en dimension trois d'espace. Nous focalisons, en particulier, sur les conditions suffisantes relatives à l'interaction mutuelle entre les champs de vitesse et de vorticité, assurant éventuellement que les solutions faibles de Leray–Hopf sont en réalité des solutions fortes (cf. Définitions 1.1 et 1.2). Observons que pour le problème bidimensionnel, l'existence globale de solutions régulières a été démontrée dans [11]. L'une des observations principales pour comprendre la différence entre les problèmes en dimension d'espace deux et trois est que le comportement de la vorticité est moins compliqué pour les écoulements dans le plan : (a) la vorticité vérifie un principe de maximum ; (b) elle ne change pas de direction ; (c) elle reste toujours orthogonale au plan du mouvement.

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Le caractère borné de la vorticité implique trivialement la régularité et ces dernières années il y a eu un intérêt considérable pour comprendre dans quelle mesure les alignements de la vorticité peuvent conduire à la régularité aussi dans le cas 3D, voir [6,1,2]. Ici, nous poursuivons cette étude dans une direction un peu différente : on se propose de comprendre si la condition $u \perp \omega$ peut impliquer la régularité du problème dans $\mathbb{R}^3$. En utilisant la formulation en termes du rotationnel (3), il est facile d'observer qu'à l'opposé, la condition $u \parallel \omega$, implique la régularité, parce que le terme de convection devient un gradient : d'autre part, l'orthogonalité de la vitesse et la pression n'a pas d'implication remarquable sur le problème de la régularité. Observons que la condition $u \perp \omega$ implique l'annulation de l'hélicité, qui est une quantité conservée par les écoulements parfaits [15]. Le résultat principal que nous démontrons est le suivant :

Théorème 1.1. *Supposons que u soit une solution faible des équations de Navier–Stokes dans le tore $\mathbb{T}$, avec $u_0 \in H^4(\mathbb{T})$. S'il existe $c_1 > 0$ telle que pour p.t. $t \in]0, T[$, pour tout $x \in \mathbb{T}$, et pour tout y assez petit on a $|u(x+y,t) \cdot \omega(x,t)| \leqslant c_1|y||u(x+y,t)||\omega(x,t)|$, alors u est une solution forte (et donc régulière) dans $]0, T[$.*

Contrairement aux autres résultats concernant les critères de nature « géométrique » pour la régularité des équations de Navier–Stokes, ici nous ne faisons pas appel aux intégrales singulières, mais seulement à l'approximation (7) par de différences finies de l'équation de la vorticité. Afin de majorer le reste dans la formule de Taylor, nous utilisons des estimations sur les solutions régulières obtenues en testant l'équation avec des puissances convenables du Laplacien. Le terme approximé $\omega_j(x,t)[u_i(x+Ke_j,t) - u_i(x,t)]/K$ (après multiplication pas $\omega_i(x,t)$ et intégration) peut être estimé facilement en utilisant l'hypothèse du Théorème 1.1.

L'hypothèse sur la régularité de la vitesse initiale et sur le type de conditions au bord peuvent être considérablement améliorées. Dans [4] nous montrons des résultats semblables avec une hypotèse H^1 naturelle sur la donnée initiale et en considérant le problème dans un domaine borné avec conditions au bord de Navier (cela est dû au fait qu'on utilise l'équation du rotationnel, qui ne semble pas pouvoir s'étudier de manière satisfaisante avec des conditions au bord de Dirichlet).

1. Introduction

In this Note we consider the initial value problem for the 3D Navier–Stokes equations with periodic boundary conditions in the torus $\mathbb{T} =]0, 2\pi[^3$

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0 & \text{in } \mathbb{T} \times]0, T], \\ \nabla \cdot u = 0 & \text{in } \mathbb{T} \times]0, T], \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}, \end{cases} \tag{1}$$

and (in order to avoid inessential complications) we assume that the external force vanishes. Most of the results of this paper hold true also in a more general context, but here we present just the simplest case and refer to the last section for further developments. We use standard notations for Lebesgue L^p and Sobolev H^s spaces; the symbol $\|\,.\|_p$ will denote the norm in L^p and for all function spaces the subscript "$_\sigma$" denotes divergence-free vector fields.

We are concerned with the well-known problem of the possible global regularity of solutions. First we recall the notion of weak solution.

Definition 1.1 *(Leray–Hopf weak solution).* We say that a vector field u belonging to $L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma)$ is a weak solution to the Navier–Stokes equations (1), if the two following conditions hold true: 1)

$$\int_0^T\!\!\int_{\mathbb{T}} \big(-u\phi_t + \nabla u \nabla\phi + (u \cdot \nabla)u\phi\big)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}} u_0(x)\,\phi(x,0)\,\mathrm{d}x,$$

for each space-periodic $\phi \in C^\infty_\sigma([0,T] \times \mathbb{T})$ satisfying $\phi(T) = 0$; and in addition 2) the energy inequality is satisfied:

$$\frac{1}{2}\big\|u(t)\big\|_2^2 + \nu\int_0^t \big\|\nabla u(\tau)\big\|_2^2\,\mathrm{d}\tau \leqslant \frac{1}{2}\|u_0\|_2^2, \quad \forall t \in [0,T]. \tag{2}$$

For initial data $u_0 \in L^2_\sigma$ existence of weak solutions can be proved for arbitrary positive times, but their possible regularity/uniqueness represent open problems. In particular, it is critical to control the behavior of $\|\nabla u(t)\|_2$.

More regular classes of solutions can be considered and of particular interest are the strong solutions.

Definition 1.2 *(Strong solution).* We say that a weak solution u is strong in $[0, T]$ if

$$\nabla u \in L^\infty\big(0, T; L^2\big) \cap L^2\big(0, T; H^1\big).$$

In addition, we say that a weak solution u is strong in $[0, T_1[$ if u is strong in $[0, T]$ for each $T < T_1$.

With initial data $u_0 \in H^1_\sigma$ it is possible to prove local-in-time existence of strong solutions, which are unique (also among weak solutions) and smooth, but the problem of their existence on arbitrary large time-intervals is still open. For all these results see Leray [13] and also Refs. [7,9,10].

In two dimensions the problem is much easier since global existence of strong solutions is known, see Ladyžhenskaya [11]. One main technical difference between the 2D and the 3D case is represented by some Gagliardo–Nirenberg inequalities, whose exponents change accordingly with the space dimension, see Lemma 1 in Ladyžhenskaya [10]. Taking a look at the problem from a more geometrical point of view one can also observe that in the 2D case the vorticity field ω has only one non-zero component, it satisfies a maximum principle, and it is orthogonal to the plane of motion (cf. (5)).

The situation in three dimensions is much more complex since the vorticity (now a vector) can change both length and direction and the physical phenomena that may occur are more difficult to be understood.

Due to the difficulties in tackling the problem of regularity for the Navier–Stokes in 3D, several "conditional" regularity results have been proved. Apart the now classical Prodi–Serrin type conditions, one possible approach may be to prove regularity under additional assumption on the velocity/vorticity mimicking the 2D setting. The first results in this direction have been obtained by asking a suitable alignment of the vorticity at neighboring points. This approach has been proposed by Constantin [5] and then rigorously linked with the regularity results by Constantin and Fefferman [6], Beirão da Veiga and Berselli [1,2] and references therein.

Next, concerning the mutual interaction of velocity and vorticity one can observe that rewriting the 3D Navier–Stokes equations in the "rotational formulation," we get

$$u_t + \omega \times u - \nu \Delta u + \nabla\left(p + \frac{1}{2}|u|^2\right) = 0 \quad \text{in } \mathbb{T} \times]0, T]. \tag{3}$$

Hence, the non-linear convection term is equal to a gradient (which vanishes when tested with divergence-free functions) for "Beltrami flows," i.e., if

$$u \parallel \omega. \tag{4}$$

It is also easy to show that if u and ω are "almost parallel," more precisely if there exists a "small enough" $\epsilon > 0$ such that

$$\big|\omega(x,t) \times u(x,t)\big| \leqslant \epsilon \big|\omega(x,t)\big|\big|u(x,t)\big|, \quad \forall (x,t) \in \mathbb{T} \times]0, T[,$$

then strong solutions exist on the whole time interval $[0, T]$. Moreover, special classes of solutions satisfying (4) are known (e.g., Trkal flows, see Berker [3, § 50]). Observe that condition (4) is "*the opposite*" to the geometric situation of the 2D case.

Beside this, coming back to the 2D case we can consider the class of exact solutions introduced by G.I. Taylor [17]. These solutions – which are called also Kolmogorov flows – have several interesting properties, see Berker [3, § 39] and are defined *via* a stream function $\psi(x, y, t) = \mathrm{e}^{kt/Re} F(x, y)$, with k an eigenvalue of the Laplacian, while F is the corresponding eigenfunction. For these 2D solutions obviously velocity and vorticity are perpendicular, but the same construction has been employed by Ross Ethier and Steinman [16] to identify classes of 3D exact solutions which (contrary to the 2D case) turn out to be Beltrami flows satisfying (4).

Based on the above considerations our main intent is to understand if a "2D-type condition" for three-dimensional flows

$$u \perp \omega \tag{5}$$

(which is clearly not satisfied by non-trivial Beltrami flows) implies regularity of the solutions. To support this claim, observe also that in the axial symmetric case without swirl (i.e. $u^\theta = 0$) the velocity field is $u = (u^r, u^\theta, u^z) = (u^r, 0, u^z)$ where θ denotes the azimuthal angle. In this case, there are several results concerning the well-posedness, dating back to Ladyžhenskaya [12] and Ukhovskii and Iudovich [18].

Well-known calculations show that the vorticity vector has only the azimuthal component different from zero $\omega = (0, \omega^\theta, 0) = (0, u^r_z - u^z_r, 0)$ and a similar situation occurs. For related results see also the paper by Mahalov, Titi, and Leibovich [14].

Moreover, condition (5) is related with the vanishing (or also smallness) of the *density of the helicity* $u\cdot\omega := u_1\omega_1 + u_2\omega_2 + u_3\omega_3$. We also recall that interesting geometrical properties connecting helicity $H(t) = \int_{\mathbb{T}} u(x,t)\cdot\omega(x,t)\,\mathrm{d}x$ and knot's theory have been proved by Moffatt [15], while recent results on the helicity are also those by Foias, Hoang, and Nicolaenko [8].

2. Main result

In this section we state the main result and give a sketch of the proof. It is interesting to observe that in the proof of the previous "geometric" criteria (cf. references in the previous section) singular integrals' theory and representation's formulae were needed in an essential way. Here, we can use a much simpler approximation by finite differences and other elementary tools.

Theorem 2.1. *Let us assume that u is a weak solution in* $[0,T]$ *of the Navier–Stokes equations, with $u_0 \in H^4(\mathbb{T})$. If there exists $c_1 > 0$ such that for a.e. $t \in]0,T[$, for all $x \in \mathbb{T}$, and for all y small enough it holds*

$$\big|u(x+y,t)\cdot\omega(x,t)\big| \leqslant c_1|y|\big|u(x+y,t)\big|\big|\omega(x,t)\big|, \tag{6}$$

then u is strong (*hence regular*) *in* $[0,T]$.

Remark 1. The above condition implies that the vorticity direction $\frac{\omega}{|\omega|}$ and the velocity direction $\frac{u}{|u|}$ respectively at point x and $x+y$ become orthogonal as a Lipschitz function, when $|y|$ tends to zero.

Proof of Theorem 2.1. Let us consider the Cauchy problem for the Navier–Stokes equations (1) and suppose that u is smooth in $]0,T^*[$, where all calculations we shall perform are justified. In particular, let $[0,T^*[$ denote the *maximal* interval of existence of the strong solution such that $u_{t=t_0} = u_0$ and let us assume *per absurdum* that $T^* < T$. Necessarily $\|\omega(t)\|_2$ must blow up when t approaches T^*, see e.g., [7,13]. By using condition (6) we shall show that the solution can be continued to a larger interval, hence the solution is smooth on $[0,T]$. We consider the vorticity equation $\partial_t\omega + (u\cdot\nabla)\omega - \nu\Delta\omega = (\omega\cdot\nabla)u$ and the main idea is to approximate ∇u in the "vortex stretching term" from the right-hand side by means of finite differences, obtaining the balance equation for an "approximate vorticity" ω^K (written in coordinates, with summation over repeated indices)

$$\frac{\partial\omega_i^K(x,t)}{\partial t} + u_j^K(x,t)\frac{\partial\omega_i^K(x,t)}{\partial x_j} - \nu\frac{\partial^2\omega_i^K(x,t)}{\partial x_m^2} = \omega_j^K(x,t)\frac{u_i^K(x+Ke_j,t) - u_i^K(x,t)}{K}, \quad \text{for some } K > 0$$

(where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$). We have not at disposal *a priori* estimates on the quantity ω^K and we need to write some exact formulae involving finite differences: we use the Taylor's expansions with Lagrange's remainder to obtain

$$\begin{aligned}&\frac{\partial\omega_i(x,t)}{\partial t} + u_j(x,t)\frac{\partial\omega_i(x,t)}{\partial x_j} - \nu\frac{\partial^2\omega_i(x,t)}{\partial x_l^2}\\ &\quad = \omega_j(x,t)\frac{u_i(x+Ke_j,t) - u_i(x,t)}{K} + \frac{K}{2}\omega_j(x,t)\frac{\partial^2 u_i(\xi_i(x,K,i),t)}{\partial x_j^2},\end{aligned} \tag{7}$$

where $\xi_i(x,K,i)$ (for $i = 1,2,3$) is a suitable point belonging to the ball centered at x and with radius K.

Taking the scalar product of (7) with ω and integrating over $\mathbb{T}$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2 + \nu\|\nabla\omega\|_2^2 \leqslant \left|\int_{\mathbb{T}} \omega_j(x,t)\frac{u_i(x+Ke_j,t) - u_i(x,t)}{K}\omega_i(x,t)\,\mathrm{d}x\right| + \frac{K}{2}\big\|D^2u\big\|_\infty \int_{\mathbb{T}} \big|\omega(x,t)\big|^2\,\mathrm{d}x.$$

The first term from the right-hand side is now estimated as follows: condition (6) implies that: a) $u(x,t)\cdot\omega(x,t)=0$; b) $|u(x+Ke_i,t)\cdot\omega(x,t)|\leqslant c_1 K|u(x+Ke_i,t)||\omega(x,t)|$, and consequently

$$\left|\int_{\mathbb{T}}\omega_i(x,t)\frac{u_j(x+Ke_i,t)-u_j(x,t)}{K}\omega_j(x,t)\,\mathrm{d}x\right|\leqslant c_1\int_{\mathbb{T}}\bigl|\omega(x,t)\bigr|\bigl|u_j(x+Ke_i,t)\bigr|\bigl|\omega(x,t)\bigr|.$$

Then, we use the translational invariance of the Lebesgue norm, the Sobolev's embedding $H^1\hookrightarrow L^6$, and the Young's inequality to infer that

$$\left|\int_{\mathbb{T}}\omega_i(x,t)\frac{u_j(x+Ke_i,t)-u_j(x,t)}{K}\omega_j(x,t)\,\mathrm{d}x\right|\leqslant\|\omega\|_2\|u\|_3\|\omega\|_6\leqslant c\|\omega\|_2\|u\|_6\|\omega\|_6\leqslant c\|\omega\|_2^2\|\nabla\omega\|_2$$
$$\leqslant\frac{\nu}{4}\|\nabla\omega\|_2^2+c\|\omega\|_2^4,$$

where c is a generic constant, independent of the solution u.

To handle the second term from the right-hand side, we use the well-known result that $u\in L^\infty(0,t;H^4)$ for $t<T^*$ (see, e.g., [7, Theorem 10.6]), and consequently $D^2u\in L^\infty([0,t]\times\mathbb{T})$. In particular, by using a well-known bootstrap argument, it is possible to show an explicit dependence of the growth of *any* norm in terms of the H^1-norm of u (here we are using the periodicity in an essential way, see Section 3). A complete proof can be found in [7, Ch. 10] and the idea is to multiply the Navier–Stokes equations by $-\Delta^s u$, with $s>\frac{3}{2}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|\Delta^{\frac{s}{2}}u\bigr\|_2^2+\nu\bigl\|\Delta^{\frac{s+1}{2}}u\bigr\|_2^2\leqslant\frac{c}{\nu}\bigl\|\Delta^{\frac{s}{2}}u\bigr\|_2^4. \tag{8}$$

Next, if u is a strong solution in $[0,T^*[$, then $\int_0^t\|\Delta u(\tau)\|_2^2\,\mathrm{d}\tau\leqslant\|\nabla u_0\|^2+\frac{c}{\nu^3}\int_0^t\|\nabla u(\tau)\|^6\,\mathrm{d}\tau:=\Omega(t)$, for each $0\leqslant t<T^*$, as it follows directly by testing (1) with $-\Delta u$ (cf. [7, Ch. 9]). Consequently, we can use Gronwall's inequality in (8) with $s=2$ to obtain:

$$\bigl\|\Delta u(t)\bigr\|_2^2+\nu\int_0^t\bigl\|\Delta^{\frac{3}{2}}u(\tau)\bigr\|_2^2\,\mathrm{d}\tau\leqslant\|\Delta u_0\|_2^2\,\mathrm{Exp}\left[\frac{c}{\nu}\Omega(t)\right].$$

Next, by using recursively the same tool with $s=3,4$ we obtain

$$\bigl\|\Delta^2u(t)\bigr\|_2^2+\nu\int_0^t\bigl\|\Delta^{\frac{5}{2}}u(\tau)\bigr\|_2^2\,\mathrm{d}\tau\leqslant\bigl\|\Delta^2u_0\bigr\|_2^2\Phi\Bigl(\nu,\|u_0\|_{H^4},\sup_{0<\sigma<t}\bigl\|\nabla u(\sigma)\bigr\|_2\Bigr),$$

where $\Phi(.\,,.)$ is a non-negative smooth function of its arguments, whose precise expression is not essential. By increasing the L^2-norm, with the L^6-norm and by using a Sobolev's embedding theorem, we finally get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2+\bigl[\nu-K\|u_0\|_{H^4}\Phi\bigl(\nu,\|u_0\|_{H^4},\|\nabla u\|_2\bigr)\bigr]\|\nabla\omega\|_2^2\leqslant c\|\omega\|_2^4.$$

We can employ Gronwall's lemma as long as the term which multiplies $\|\nabla\omega\|_2^2$ is non-negative. This is surely the case in some interval $[0,\delta]$, by choosing the arbitrary $K>0$ small enough (recall that for strong solutions the map $t\mapsto u(t)$ is continuous with values in H^1). This argument and the application of the energy inequality (2), implies

$$\sup_{0<t<\delta}\bigl\|\omega(t)\bigr\|_2^2\leqslant c\|\omega_0\|_2^2\,\mathrm{Exp}\left[\int_0^\delta\bigl\|\nabla u(\tau)\bigr\|^2\,\mathrm{d}\tau\right]\leqslant c\|\omega_0\|_2^2\,\mathrm{Exp}\left[\frac{\|u_0\|^2}{2\nu}\right].$$

Since the above bound is independent of δ, we can estimate the supremum of Φ and choose accordingly a small $K>0$ in order to use the same reasoning on the whole interval $[0,T^*[$. This finally implies an uniform bound for the L^2-norm of the vorticity on $[0,T^*[$ and contradicts the fact that T^* is an "*époque de irrégularité*" (cf. Leray [13]) concluding the proof. □

3. Further results

The hypotheses (on the boundary and initial conditions) of Theorem 2.1 can be considerably improved. We preferred to present a simpler case since the proof is much shorter and without technical difficulties. By using the same ideas and some other tools (especially a more sophisticated bootstrap argument with time weights) it is possible to prove the following result, see [4]:

Theorem 3.1. *Let us consider the* 3D *Navier–Stokes equations in a smooth bounded domain, supplemented with the Navier boundary conditions. Let us assume that u is a weak solution of the Navier–Stokes equations, with* $u_0 \in H^1_\sigma$. *If there exists either*

(a) $c_1 > 0$ *such that for a.e.* $t \in]0, T[$ *and for all* $x \in \mathbb{R}^3$ *and for all* y *small enough*

$$\left|u(x+y,t)\cdot\omega(x,t)\right| \leqslant c_1|y|\left|u(x+y,t)\right|\left|\omega(x,t)\right|;$$

(b) *a "small"* $c_2 = c_2(\nu, \|u_0\|_{H^1}) > 0$ *such that for a.e.* $t \in]0, T[$ *and for all* $x \in \mathbb{R}^3$ *and for all* y *small enough*

$$\left|u(x+y,t)\cdot\omega(x,t)\right| \leqslant c_2\left|u(x+y,t)\right|\left|\omega(x,t)\right|,$$

then u is strong, hence regular in $[0, T]$.

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