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Non-existence for travelling waves with small energy for the Gross–Pitaevskii equation in dimension $N \geqslant 3$

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Abstract

We prove that the Ginzburg–Landau energy of non-constant travelling waves of the Gross–Pitaevskii equation has a lower positive bound, depending only on the dimension, in any dimension larger or equal to three. In particular, we conclude that there are no non-constant travelling waves with small energy. ***To cite this article: A. de Laire, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Non existence pour les ondes progressives d'énergie petite pour l'équation de Gross–Pitaevskii en dimension $N \geqslant 3$. On démontre que l'énergie de Ginzburg–Landau des ondes progressives non constantes de l'équation de Gross–Pitaevskii est bornée inférieurement par une constante positive qui ne dépend que de la dimension, pour toute dimension supérieure ou égale à trois. En particulier, on en déduit qu'il n'existe pas d'onde progressive non constante d'énergie petite. ***Pour citer cet article : A. de Laire, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

On s'intéresse aux ondes progressives non constantes d'énergie finie pour l'équation de Gross–Pitaevskii $i\partial_t\Psi = \Delta\Psi + \Psi(1-|\Psi|^2)$ dans $\mathbb{R}^N \times \mathbb{R}$, en dimension $N \geqslant 3$. Les ondes progressives pour cette équation sont des solutions de la forme $\Psi(x,t) = v(x_1 - ct, x_\perp)$, $x_\perp = (x_2,\ldots,x_N)$, où la fonction v vérifie l'équation

$$ic\partial_1 v + \Delta v + v(1-|v|^2) = 0 \quad \text{dans } \mathbb{R}^N. \tag{1}$$

Grâce aux résultats de Gravejat [3], on peut supposer que la vitesse c de l'onde progressive est telle que $0 < c \leqslant \sqrt{2}$. Le Hamiltonien associé à (1) est l'énergie de Ginzburg–Landau donnée par

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|v|^2)^2.$$

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Tarquini a montré dans [9] l'existence d'une valeur minimale $\mathcal{E}(N,c)$ pour l'énergie de Ginzburg–Landau des ondes solitaires, qui ne dépend que de la dimension N et de la vitesse c, ce qui implique que les seules solutions possibles de (1) de vitesse c avec une énergie plus petite que $\mathcal{E}(N,c)$ sont les constantes. Béthuel, Gravejat et Saut [1] ont amélioré ce résultat en dimension trois, en démontrant qu'il existe une énergie minimale $\bar{\mathcal{E}}$ indépendante de c. Dans cette Note on montre qu'il est possible d'étendre ce dernier résultat pour toute dimension $N \geqslant 3$. Plus précisément,

Théorème 0.1. *Soit $N \geqslant 3$. Il existe une constante positive $\mathcal{E}(N)$, qui ne dépend que de N, telle que pour toute solution non constante v de* (1)*, on ait $E(v) \geqslant \mathcal{E}(N)$. En particulier, il n'existe pas de solution non constante pour* (1) *d'énergie petite.*

1. Introduction

The Gross–Pitaevskii equation $\mathrm{i}\partial_t\Psi = \Delta\Psi + \Psi(1-|\Psi|^2)$ on $\mathbb{R}^N \times \mathbb{R}$, whose Hamiltonian is the Ginzburg–Landau energy given by

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\Psi|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \left(1-|\Psi|^2\right)^2,$$

appears as a relevant model in several areas of physics: superfluidity, superconductivity, non-linear optics and the Bose–Einstein condensation (see e.g. [4–6,8]). In this work, we investigate the energy of travelling waves to this equation, i.e. solutions of the form $\Psi(x,t) = v(x_1 - ct, x_\perp)$, $x_\perp = (x_2,\dots,x_N)$. Here, the parameter $c \in \mathbb{R}$ corresponds to the speed of the travelling waves. Using complex conjugation, we may restrict to the case $c \geqslant 0$. The equation for the profile v is given by

$$\mathrm{i}c\partial_1 v + \Delta v + v\left(1-|v|^2\right) = 0 \quad \text{on } \mathbb{R}^N. \tag{2}$$

2. Main result

A result of Tarquini [9] states that there exists a minimal value $\mathcal{E}(N,c)$ for the Ginzburg–Landau energy of travelling waves, depending only on N and c. This lower bound for the energy functional implies that non-constant finite energy solutions of (2) of sufficiently small energy, with respect to their speed, are excluded in dimension $N \geqslant 2$. Furthermore $\mathcal{E}(N,c) \to 0$ as $c \to \sqrt{2}$. This result has been recently improved by Béthuel, Gravejat and Saut [1] in dimension three, proving that there exists some universal positive bound for the energy functional for non-constant travelling waves.

Our aim is to extend the result of Béthuel, Gravejat and Saut [1] in any dimension larger than three, and therefore also to improve the non-existence theorem of Tarquini [9]. More precisely, our main result is

Theorem 2.1. *Let $N \geqslant 3$. There exists some positive constant $\mathcal{E}(N)$, depending only on N, such that any non-constant finite energy solution v of* (2) *satisfies $E(v) \geqslant \mathcal{E}(N)$. In particular, there are no non-constant solutions of* (2) *with small energy.*

3. Proof of main result

In dimension $N \geqslant 3$, it follows from [3] that the speed of non-constant finite energy solutions of (2) satisfy $0 < c \leqslant \sqrt{2}$. From Lemma 3 in [9], we deduce that $\|1-|v|^2\|_{L^\infty(\mathbb{R}^N)} \leqslant K(N)E(v)^{\frac{1}{2(N+1)}}$, where $K(N)$ is a positive constant, depending only on N. Therefore, choosing a possibly smaller constant $\mathcal{E}(N)$, we may assume that v satisfies

$$\inf\left\{\left|v(x)\right|, x \in \mathbb{R}^N\right\} \geqslant \frac{1}{2}. \tag{3}$$

We recall that v is a smooth function (see e.g. [2]), and then in view of (3), v may be expressed as $v = \rho e^{\mathrm{i}\varphi}$, where ρ and φ are scalar functions, and φ is defined modulo a multiple of 2π. Defining also the quantity $\eta = 1-\rho^2$, we have

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_1^2\eta = -2\Delta\left(|\nabla v|^2 + \eta^2 - c\eta\partial_1\varphi\right) - 2c\partial_1 \operatorname{div}(\eta\nabla\varphi). \tag{4}$$

Applying the Fourier transform to (4), we obtain

$$\hat{\eta}(\xi) = L_c(\xi)\widehat{F}(\xi), \tag{5}$$

where

$$\widehat{F}(\xi) = 2\widehat{R_0}(\xi) - 2c\sum_{j=2}^{N}\frac{\xi_j^2}{|\xi|^2}\widehat{R_1}(\xi) + 2c\sum_{j=2}^{N}\frac{\xi_1\xi_j}{|\xi|^2}\widehat{R_j}(\xi), \tag{6}$$

$R_0 = |\nabla v|^2 + \eta^2$, $R_j = \eta\partial_j\varphi$, $j \in \{1, \dots, N\}$, and

$$L_c(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}. \tag{7}$$

Now we recall two results of Béthuel, Gravejat and Saut. The first one corresponds to Lemma 2.9 in [1], and the second one is an immediate extension to $\mathbb{R}^N$ of some part of the argument used in Lemma 2.15 (see inequality (2.65) in [1]).

Lemma 3.1. *Let v be a non-constant finite energy solution to* (2) *satisfying* (3). *Then,*

$$E(v) \leqslant 7c^2\|\eta\|_{L^2(\mathbb{R}^N)}^2.$$

Lemma 3.2. *For any $1 < q < \infty$, there exists a positive constant $K(N,q)$, depending only on N and q, such that*

$$\|F\|_{L^q(\mathbb{R}^N)} \leqslant K(N,q)E(v)^{\frac{1}{q}}.$$

We denote $\mathcal{L}_c$ the operator given by $\widehat{\mathcal{L}_c(f)} = L_c\hat{f}$, $\forall f \in S(\mathbb{R}^N)$. We recall that in the case that there exists a constant K such that $\|\mathcal{L}_c(f)\|_{L^q(\mathbb{R}^N)} \leqslant K\|f\|_{L^p(\mathbb{R}^N)}$, L_c is called a Fourier multiplier from L^p to L^q. We notice that identity (5) implies that η is the value of the multiplier operator associated to L_c, evaluated in the function F given by (6), that is

$$\mathcal{L}_c(F) = \eta. \tag{8}$$

In order to complete the proof of Theorem 2.1, we need the following lemma, whose proof we postpone to the next section:

Lemma 3.3. *Let $c \in (0, \sqrt{2}]$. For any $\frac{2}{2N-1} \leqslant \alpha \leqslant \frac{2}{N}$ and $\frac{1}{1-\alpha} < q < \infty$, L_c given by* (7) *is a Fourier multiplier from L^p to L^q, with $\frac{1}{p} = \frac{1}{q} + \alpha$. More precisely, there exists a positive constant $K(N,\alpha,q)$, depending only on N, α and q, such that*

$$\left\|\mathcal{L}_c(f)\right\|_{L^q(\mathbb{R}^N)} \leqslant K(N,\alpha,q)\|f\|_{L^p(\mathbb{R}^N)}, \quad \forall f \in L^p\big(\mathbb{R}^N\big). \tag{9}$$

In view of (8), applying Lemma 3.3, with $\alpha = \frac{2}{2N-1}$ and $q = 2$, we deduce that there exists a positive constant $K(N)$, depending only on N, such that

$$\|\eta\|_{L^2(\mathbb{R}^N)} \leqslant K(N)\|F\|_{L^{\frac{2(2N-1)}{2N+3}}(\mathbb{R}^N)}. \tag{10}$$

Combining Lemmas 3.1, 3.2 and (10), we conclude that

$$E(v) \leqslant 7c^2\|\eta\|_{L^2(\mathbb{R}^N)}^2 \leqslant 7c^2K(N)E(v)^{\frac{2N+3}{2N-1}}. \tag{11}$$

Since $c \in (0, \sqrt{2}]$, inequality (11) implies that $E(v) \geqslant (14K(N))^{\frac{1-2N}{4}}$, which finishes the proof of Theorem 2.1.

4. Proof of Lemma 3.3

Here we use the standard multi-index notation, i.e. if $k=(k_1,\dots,k_N)\in\mathbb{N}^N$, $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N$ then $D^k=\partial_{\xi_1}^{k_1}\cdots\partial_{\xi_N}^{k_N}$, $|k|=\sum_{j=1}^N k_j$ and $\xi^k=\prod_{j=1}^N \xi_j^{k_j}$.

Lemma 4.1. *Let $c\in(0,\sqrt{2}]$. For any $k=(k_1,\dots,k_N)\in\{0,1\}^N$, $m=|k|$, $1\leqslant m\leqslant N$, L_c is a smooth function on $\mathbb{R}^N\backslash\{0\}$ and*

$$D^k L_c(\xi)=\frac{\xi^k}{(|\xi|^4+2|\xi|^2-c^2\xi_1^2)^{m+1}}P_{m,c}\big(|\xi|^2,\xi_1^2\big), \tag{12}$$

where $P_{m,c}$ is a two-variable polynomial of degree $m+1$. More precisely, for $x,y\in\mathbb{R}$,

$$P_{m,c}(x,y)=\gamma_m(c)x^{m+1}+\sum_{1\leqslant i+j\leqslant m}^{m}\gamma_{m,i,j}(c)x^i y^j, \tag{13}$$

where $\{\gamma_{m,i,j}\}_{i,j=1}^m$ and γ_m are polynomial functions of the variable c. Furthermore, in the case $k_1=1$, setting $\alpha_m=\gamma_{m,1,0}$, $\beta_m=\gamma_{m,0,1}$ and $\lambda_m(c)=\frac{\alpha_m(c)+\beta_m(c)}{2-c^2}$, we have

$$\alpha_m(c)=(-1)^{m+1}2^{2m-1}(m-1)!c^2, \tag{14}$$

$$\beta_m(c)=(-1)^{m+1}2^{2m-2}(m-1)!c^2\big(c^2(n-1)-2n\big), \tag{15}$$

$$\lambda_m(c)=(-1)^{m+1}2^{2m-2}(m-1)!(m-1)c^2. \tag{16}$$

In particular, λ_m is a well defined and bounded function on $(0,\sqrt{2})$.

Proof. The differentiability of L_c is immediate. The case $m=1$ is checked explicitly, since we have

$$\partial_i L_c(\xi)=\frac{2\xi_i}{(|\xi|^4+2|\xi|^2-c^2\xi_1^2)^2}\big(-|\xi|^4-c^2\xi_1^2+c^2\delta_{1,i}|\xi|^2\big). \tag{17}$$

We fix now m, with $1<m\leqslant N$. Let us suppose that (12) and (13) are valid for some $1\leqslant n<m$. We take any $r=(r_1,\dots,r_N)\in\{0,1\}^N$ such that $|r|=n+1$ and define $j^*=\max\{1\leqslant j\leqslant N\mid r_j=1\}$. Then $j^*>1$, and we consider $\tilde r=(\tilde r_1,\dots,\tilde r_N)\in\{0,1\}^N$ given by $\tilde r_i=r_j(1-\delta_{i,j^*})$, $i,j\in\{1,\dots,N\}$. Therefore, $|\tilde r|=n$ and we have,

$$D^r L_c(\xi)=\partial_{j^*}^1\big(\partial_1^{\tilde r_1}\partial_2^{\tilde r_2}\dots\partial_N^{\tilde r_N}L_c\big)(\xi)=\frac{\xi^r}{(|\xi|^4+2|\xi|^2-c^2\xi_1^2)^{n+2}}P_{n+1,c}\big(|\xi|^2,\xi_1^2\big),$$

where

$$P_{n+1,c}\big(|\xi|^2,\xi_1^2\big)=2\partial_x P_{n,c}\big(|\xi|^2,\xi_1^2\big)\big(|\xi|^4+2|\xi|^2-c^2\xi_1^2\big)-(n+1)\big(4|\xi|^2+4\big)P_{n,c}\big(|\xi|^2,\xi_1^2\big). \tag{18}$$

Using this inductive argument, we conclude the first part of the lemma, that is, identities (12) and (13). In order to deduce, in the case $k_1=1$, that the coefficients of lower terms are explicitly given by (14) and (15), we use the same inductive argument but we replace the polynomial expression (18) by the following one

$$P_{n+1,c}(x,y)=\bar\gamma_n(c)x^{n+2}+\sum_{\substack{i,j=0\\2\leqslant i+j\leqslant n}}^{n+1}\bar\gamma_{n,i,j}(c)x^i y^j-4n\alpha_n(c)x-\big(2c^2\alpha_n(c)+4(n+1)\beta_n(c)\big)y,$$

for some $\{\bar\gamma_{n,i,j}\}_{i,j=1}^n$, $\bar\gamma_n$, polynomial functions of the variable c. The formulas (14) and (15) allow us to finish the induction. Finally we notice that identity (16) is an immediate consequence of (14) and (15). □

An important property that follows from identities (14)–(16) is that for small values of ξ, we may compute an explicit bound for $P_{m,c}$, that is

Lemma 4.2. *For any $c\in[0,\sqrt{2}]$ and $0<|\xi|\leqslant 1$, $k=(1,k_2,\dots,k_N)$, $m=|k|$, we have $|P_{m,c}(|\xi|^2,\xi_1^2)|\leqslant K(N)(|\xi|^4+2|\xi|^2-c^2\xi_1^2)$, where $K(N)$ is a positive constant depending only on N.*

Proof. The only delicate terms of $P_{m,c}$ to estimate are the ones associated to $|\xi|^2$ and ξ_1^2, this is $\alpha_m(c)|\xi|^2+\beta_m(c)\xi_1^2$. Indeed, the other terms of $P_{m,c}(|\xi|^2,\xi_1^2)$ are easily bounded by $K(N)|\xi|^4$, for some constant $K(N)$ depending only on N. For example,

$$\left|\gamma_{m,1,1}(c)\right|\xi_1^2|\xi|^2 \leqslant \frac{1}{2}\|\gamma_{m,1,1}\|_{L^\infty[0,\sqrt{2}]}\left(\xi_1^4+|\xi|^4\right)\leqslant K(N)|\xi|^4 \leqslant K(N)\left(|\xi|^4+2|\xi|^2-c^2\xi_1^2\right),$$

where we used that the L^∞-norm in $[0,\sqrt{2}]$ of the functions $\gamma_{m,i,j}$ only depends on the dimension. Next we derive the bound for $\alpha_m(c)|\xi|^2+\beta_m(c)\xi_1^2$. Denoting $\xi=r\sigma$, where $0<r\leqslant 1$ and $\sigma=(\sigma_1,\sigma_\perp)\in\mathbb{S}^{N-1}$, this is equivalent to prove that

$$\exists K>0,\ \forall c\in[0,\sqrt{2}],\ \forall\sigma_1\in[0,1],\ \forall r\in(0,1],\quad \left|\alpha_m(c)+\sigma_1^2\beta_m(c)\right|\leqslant K\left(r^2+2-c^2\sigma_1^2\right). \tag{19}$$

Using the continuity of α_m and β_m, inequality (19) automatically follows from

$$\exists K>0,\ \forall c\in[0,\sqrt{2}),\ \forall\rho\in[0,1],\quad \left|\alpha_m(c)+\rho\beta_m(c)\right|\leqslant K\left(2-c^2\rho\right). \tag{20}$$

We shall prove (20) arguing by contradiction. If (20) were false, there would exist sequences

$$K_n\to\infty,\quad c_n\in[0,\sqrt{2}),\quad c_n\to\bar{c}\in[0,\sqrt{2}],\quad \rho_n\to\bar{\rho}\in[0,1], \tag{21}$$

such that

$$\left|\alpha_m(c_n)+\rho_n\beta_m(c_n)\right|>K_n\left(2-c_n^2\rho_n\right)\geqslant 0. \tag{22}$$

In particular,

$$\lim_{n\to\infty}\frac{2-c_n^2\rho_n}{|\alpha_m(c_n)+\rho_n\beta_m(c_n)|}=0. \tag{23}$$

From the continuity of α_m and β_m, the denominator in (23) is bounded, so that (23) implies $\bar{c}^2\bar{\rho}=2$, and hence $\bar{c}=\sqrt{2}$ and $\bar{\rho}=1$. Setting $\varepsilon_n=1-\rho_n$ and $s_n=\frac{\varepsilon_n}{2-c_n^2}$, we write

$$\frac{2-c_n^2\rho_n}{|\alpha_m(c_n)+\rho_n\beta_m(c_n)|}=\frac{1+s_nc_n^2}{|\lambda_m(c_n)-s_n\beta_m(c_n)|}. \tag{24}$$

Passing possibly to a subsequence, $s_n\to\bar{s}$, with $\bar{s}\in[0,\infty]$. We note from Lemma 4.1 that β_m and λ_m are bounded functions of c. If $\bar{s}\in[0,\infty)$, we take the limit in (24), so that in view of (23), we deduce that $\bar{s}=-\frac{1}{2}$, which is a contradiction. We may handle the case $\bar{s}=\infty$ in a similar way, with the difference that we first divide the numerator and the denominator of the r.h.s. of (24) by s_n. Then passing to the limit, we deduce that $\bar{c}=0$, which gives us again a contradiction. □

Now we are able to deduce an uniform bound (with respect to the speed) for L_c:

Proposition 4.1. *Let $c\in(0,\sqrt{2}]$ and $k=(k_1,k_2,\dots,k_N)\in\{0,1\}^N$, with $|k|\leqslant N$. Then for any $|\xi|\geqslant 1$,*

$$\left|D^kL_c(\xi)\right|\leqslant\frac{K(N)}{|\xi|^{|k|+2}}, \tag{25}$$

and for any $0<|\xi|\leqslant 1$,

$$\left|D^kL_c(\xi)\right|\leqslant\frac{K(N)|\xi^k|}{(|\xi|^4+2|\xi|^2-c^2\xi_1^2)^{|k|+1}}\left((1-k_1)|\xi|^2+k_1\left(|\xi|^4+2|\xi|^2-c^2\xi_1^2\right)\right), \tag{26}$$

where $K(N)$ is a constant depending only on N.

Proof. From (12) and (13), with $m=|k|$, we conclude that for any $|\xi|\geqslant 1$,

$$\left|D^kL_c(\xi)\right|\leqslant K(N)|\xi|^{-3m-4}\left|P_{m,c}\left(|\xi|^2,\xi_1^2\right)\right|\leqslant K(N)|\xi|^{-3m-4}|\xi|^{2(m+1)},$$

which proves (25). To derive (26), we note that in view of (12), it is enough to prove that for any $0<|\xi|\leqslant 1$,

$$\left|P_{m,c}\left(|\xi|^2,\xi_1^2\right)\right|\leqslant K(N)\left((1-k_1)|\xi|^2+k_1\left(|\xi|^4+2|\xi|^2-c^2\xi_1^2\right)\right). \tag{27}$$

If $k_1=0$, inequality (27) is trivial. In the case $k_1=1$, this bound corresponds exactly to Lemma 4.2. □

Proof of Lemma 3.3. Firstly, we notice that the condition $N \geqslant 3$ implies $0 < \alpha < 1$, so that the set of valid pairs $p \geqslant 1$ and $q \geqslant 1$ is not empty. From Proposition 4.1 we conclude that, for any $|\xi| \geqslant 1$, $k = (k_1, \dots, k_N) \in \{0,1\}^N$, $|k| \leqslant N$,

$$\prod_{j=1}^{N} |\xi_j|^{\alpha + k_j} \left| D^k L_c(\xi) \right| \leqslant \frac{K(N)}{|\xi|^{2 - N\alpha}} \leqslant K(N), \tag{28}$$

provided that $\alpha \leqslant \frac{2}{N}$, for some constant $K(N)$ depending only on N. On the other hand, if $0 < |\xi| \leqslant 1$, we set $\xi = r\sigma$, with $r > 0$ and $\sigma = (\sigma_1, \sigma_\perp) \in \mathbb{S}^{N-1}$. Then we have that $|\xi_j| \leqslant r|\sigma_\perp|$, for any $j \in \{2, \dots, N\}$, and also that $|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2 \geqslant r^2(r^2 + 2\sigma_\perp^2)$, for any $c \in (0, \sqrt{2}]$. From (26), we conclude that

$$\begin{aligned} \prod_{j=1}^{N} |\xi_j|^{\alpha + k_j} \left| D^k L_c(\xi) \right| &\leqslant K(N) \frac{r^{2(|k| - k_1 + 1) + \alpha N} |\sigma_\perp|^{\alpha(N-1) + 2(|k| - k_1)}}{r^{2(|k| - k_1 + 1)} (r^2 + 2\sigma_\perp^2)^{|k| - k_1 + 1}} \\ &\leqslant K(N) \max\{r, |\sigma_\perp|\}^{\alpha(2N-1) - 2} \leqslant K(N), \end{aligned} \tag{29}$$

for any $k = (k_1, \dots, k_N) \in \{0,1\}^N$, $|k| \leqslant N$, on condition that $\alpha \geqslant \frac{2}{2N-1}$.

Finally, from (28) and (29) we have that for every $\frac{2}{2N-1} \leqslant \alpha \leqslant \frac{2}{N}$,

$$\sup\left\{ \left| \xi_1^{k_1 + \alpha} \cdots \xi_1^{k_n + \alpha} D^k L_c(\xi) \right|,\ \xi \in \mathbb{R}^N \backslash \{0\},\ k \in \{0,1\}^N, |k| \leqslant N \right\} \leqslant K(N),$$

and therefore Lemma 3.3 is now an immediate consequence of Lizorkin's multiplier theorem (see e.g. [7]). □

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