



Differential Geometry

Differentiability of the isoperimetric profile and topology of analytic Riemannian manifolds [☆]

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Abstract

We show that smooth isoperimetric profiles are exceptional for real analytic Riemannian manifolds. For instance, under some extra assumptions, this can happen only on topological spheres. **To cite this article:** R. Grimaldi et al., *C. R. Acad. Sci. Paris, Ser. I 347 (2009)*.

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Résumé

Différentiabilité du profil isopérimétrique et topologie des variétés riemanniennes analytiques réelles. On montre que la différentiabilité du profil isopérimétrique est une condition très contraignante pour les variétés riemanniennes analytiques réelles. Par exemple, sous une hypothèse supplémentaire, ce n'est possible que si la variété est homéomorphe à une sphère. **Pour citer cet article :** R. Grimaldi et al., *C. R. Acad. Sci. Paris, Ser. I 347 (2009)*.

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Le profil isopérimétrique d'une variété riemannienne M est la fonction I qui, en $v \in (0, \text{vol}(M))$, vaut la borne inférieure $I(v)$ des volumes des bords des domaines de volume égal à v .

Il y a peu de variétés riemanniennes dont on connaisse exactement le profil (voir [1] pour un état de l'art). Parmi tous les exemples connus, seule la sphère ronde a un profil lisse. Toute variété possède-t-elle une métrique riemannienne de profil lisse ? Dans [3], on construit sur toute variété de dimension 2 ayant suffisamment de bouts des métriques lisses de profil lisse. Il est probable que des exemples similaires existent sur de nombreuses variété compactes. En revanche, il semble plus difficile de construire des exemples analytiques réels.

On sait [2] que le maximum de la courbure scalaire intervient dans le développement asymptotique du profil près de 0. Une condition suffisante pour que le profil soit lisse au voisinage de 0 est que ce maximum soit atteint en un

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seul point, et que celui-ci soit un point critique non dégénéré, voir [6]. Sous cette hypothèse supplémentaire, nous montrons que demander un profil lisse est très restrictif.

Théorème 0.1. *Soit M une variété riemannienne compacte analytique réelle. On fait les hypothèses suivantes.*

- (i) *Toutes les solutions du problème isopérimétrique dans M sont des domaines de bord lisses.*
- (ii) *La courbure scalaire atteint son maximum en un seul point, et celui-ci est un point critique non dégénéré.*
- (iii) *Le profil isopérimétrique de M est lisse.*

Alors M est homéomorphe à une sphère.

Remarque 1. L'hypothèse (i) est toujours satisfaite si $\dim(M) < 8$. Nous ignorons si on peut se passer de l'hypothèse (ii).

Preuve. Si le profil est lisse, il est analytique. Les solutions lisses du problème isopérimétrique forment alors un espace analytique réel de dimension finie. Par hypothèse, les solutions de petit volume forment un unique arc, une famille à un paramètre (on peut prendre le volume comme paramètre) de domaines proches des petites boules centrées au maximum de la courbure scalaire. Cet arc analytique se prolonge analytiquement, et ne peut se terminer que par la famille des domaines dont les précédents sont les complémentaires. M est donc partagée en deux domaines difféomorphes à des boules fermées. \square

1. Introduction

The *isoperimetric profile* of a Riemannian manifold M is the function I which maps $v \in (0, \text{vol}(M))$ to the infimum $I(v)$ of boundary volumes of domains of volume equal to v .

Very few profiles of compact Riemannian manifolds are exactly known ([1] contains a state of the art). Among those, only the round sphere has a smooth profile. We address here the differentiability question: does every compact manifold admit a metric with smooth profile? In [3], smooth metrics with smooth profiles are constructed on 2-dimensional manifolds, provided they have enough ends. It is likely that similar examples can be found on many compact manifolds. However, it seems much harder to produce real analytic metrics with smooth profiles.

It is known [2] that the asymptotics of the isoperimetric profile near 0 is determined by the maximum of scalar curvature. A sufficient condition for the profile to be smooth near 0 is that this maximum be achieved at only one point, which is a nondegenerate critical point, see [6]. Under this assumption, we show that having a smooth profile is very restrictive.

Theorem 1.1. *Let M be a compact real analytic Riemannian manifold. Assume the following:*

- (i) *All solutions of the isoperimetric problem in M are domains with smooth boundary;*
- (ii) *The scalar curvature function achieves its maximum at a unique point, and this point is a nondegenerate critical point;*
- (iii) *The isoperimetric profile of M is smooth.*

Then M is homeomorphic to a sphere.

Remark 1. Assumption (i) is always satisfied if $\dim(M) < 8$. We do not know whether assumption (ii) can be removed.

2. Real analytic spaces

We collect more or less standard material on real analytic spaces. We express thanks to Jean-Jacques Risler for his help with these matters.

2.1. Definitions

The following definitions are borrowed from H. Hironaka, [5], Sections 1 and 5:

Definition 2.1. (H. Hironaka, [5], Definition 1.1.) A *ringed space* is a topological space equipped with a sheaf of commutative algebras over \mathbf{R} . A *morphism* of ringed spaces $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is the data of a continuous map $f : X \rightarrow Y$ together with a homomorphism of sheaves $f_* : \mathcal{O}_X \rightarrow \mathcal{O}_Y$. The stalk $\mathcal{O}_{X,x}$ at each point is a local ring. Its Krull dimension (the length of the longest chain of prime ideals in the maximal ideal) is called the *dimension of X at x* .

Example 1. Let $U \subset \mathbf{R}^n$ be an open set, let $g : U \rightarrow \mathbf{R}^m$ be a real analytic map. Then $U \cap g^{-1}(0)$ equipped with the sheaf of restrictions (to open subsets of $U \cap g^{-1}(0)$) of real analytic functions on open subsets of \mathbf{R}^n , is a ringed space. Such ringed spaces are called *local models*.

Definition 2.2. (H. Hironaka, [5], Definition 5.6.) A local model is *smooth* at a point x if the defining map g is a submersion at x .

Definition 2.3. (H. Hironaka, [5], Definition 1.5.) A *real analytic space* is a ringed space in which every point has a neighborhood isomorphic to some local model. Its dimension is the supremum of dimensions at points. Smooth points are those where the local model is smooth.

2.2. Extracting a 1-dimensional subset

Proposition 1. Let X be a n -dimensional paracompact real analytic space. Then X admits a locally finite filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ by closed real analytic subspaces X_i such that $X_i \setminus X_{i-1}$ is smooth of dimension i .

Proof. The proof is an adaptation of Proposition 5.8 of [5]. X admits a complexification $X^{\mathbf{C}}$. This is a complex analytic space containing X equipped with an antiholomorphic involution σ whose fixed point set equals X . The set $X_{n-1}^{\mathbf{C}}$ of points of $X^{\mathbf{C}}$ where $X^{\mathbf{C}}$ is not smooth and n -dimensional is defined in local models by holomorphic equations. Therefore $X_{n-1}^{\mathbf{C}}$ is a complex analytic subspace. Let $X_{n-1} = X \cap X_{n-1}^{\mathbf{C}}$. Then $X \setminus X_{n-1}$ is smooth of dimension n . Define other X_i 's recursively. \square

Corollary 2. Let X be a paracompact real analytic space. Let x be a smooth point where X has dimension 1. There exists a closed real analytic subset $Y \subset X$ which contains a neighborhood of x and whose dimension at each point is 1.

Proof. Put $Y = X_1$. By construction of X_{n-1}, \dots, X_1 , the open set U of points where X is smooth of dimension 1 is contained in Y . \square

2.3. Endpoints

Proposition 3. (See D. Sullivan, [7], page 166, example a.) Let X be a 1-dimensional real analytic space. Let $x \in X$ be a singular point. Then a punctured neighborhood of x in X consists of an even number of arcs. They match pairwise into real analytic arcs.

Corollary 4. Let X be a real analytic space whose dimension at each point is 1. Assume that the singular set of X is compact. Then each connected component of X has a finite, even number of ends.

Proof. Let $S \subset X$ denote the singular set of X . Since S is a closed real analytic subspace, S is discrete, and therefore finite. Since the property of being a real analytic space is local, and local models are locally connected, one can assume that X is connected. $X \setminus S$ is a 1-dimensional manifold, thus a collection of arcs and circles. Since X is connected, either S is empty, or $X \setminus S$ contains a circle (and then X is a circle), or there are no circles in $X \setminus S$. Then, when S is

not empty, each arc has at least one endpoint in S , thus there are finitely many arcs. Ends correspond to ends of arcs which do not belong to S . Let N denote the number of ends of X and P the number of pairs (s, a) where $s \in S$ and a is an arc with s as an endpoint. Then $N + P$ is twice the number of arcs, P is even (according to Proposition 3), thus N is even. \square

3. Constant mean curvature hypersurfaces

Definition 3.1. Fix $\alpha \in (0, 1)$. Let us define the $C^{2,\alpha}$ -topology on the space of domains with smooth boundary as follows: as neighborhoods of a smooth domain B , take all domains S whose boundary is the graph, in normal exponential coordinates, of a $C^{2,\alpha}$ -small function on ∂B .

Proposition 3.2. Let M be a compact real analytic Riemannian manifold. Let f be a positive real analytic function on the interval $(0, \text{vol}(M))$. The set X^f of pairs (B, h) where B is a smooth domain whose boundary has constant mean curvature equal to h and satisfying $\text{vol}(\partial B) = f(\text{vol}(B))$ has a natural structure of a finite dimensional real analytic space.

Proof. The proof is essentially contained in Section 4 of [4]. Let (B, k) belong to X^f . By definition, a neighborhood of B consists of domains D_u whose boundaries are graphs in normal coordinates of $C^{2,\alpha}$ functions u which satisfy the constant mean curvature equation $H(u) = h$, $h \in \mathbf{R}$, i.e. $D_u = \{\exp_x(u(x)v(x)) : x \in \partial B\}$, where v is the unit normal vectorfield on ∂B . Since the linearized operator at $u = 0$, L_B , need not be invertible, one solves instead

$$\Phi_B(u, h) = P_B(H_B(u) - h) = 0,$$

where P_B is the orthogonal projection onto the L^2 -orthogonal complement K_B^\perp of the kernel of L_B in $C^{0,\alpha}(\partial B)$. Then

$$\Phi_B : C^{2,\alpha}(\partial B) \times \mathbf{R} \rightarrow K_B^\perp$$

is a real analytic map, whose linearization at 0 in the direction of $C^{2,\alpha}(\partial B)$ is $P_B \circ L_B$. By construction, it is onto with a finite dimensional kernel. Therefore the set of solutions is a finite dimensional smooth real analytic submanifold Y_B of $C^{2,\alpha}(\partial B) \times \mathbf{R}$ in a neighborhood of B . On Y_B ,

$$(u, h) \mapsto (H(u) - h, \text{vol}(\partial D_u) - f(\text{vol}(D_u)))$$

is a real analytic map, whose zero set, denoted by X_B , is a neighborhood of B in X^f . A choice of coordinates for Y_B maps X_B to a local model. The sheaf \mathcal{O}_{X^f} is defined on small enough open sets by restricting to X_B analytic functions on $C^{2,\alpha}(\partial B) \times \mathbf{R}$. Different choices of B yield local embeddings of X^f into Banach spaces which differ only up to a change of chart for the (infinite dimensional) manifold structure on the space of domains. Since constant mean curvature hypersurfaces in real analytic Riemannian manifolds are real analytic, these changes of charts are real analytic diffeomorphisms between open sets of $C^{2,\alpha}$ spaces. This yields the structure of a real analytic space on X^f . \square

4. Compactness

Proposition 4.1. Let M be a compact Riemannian manifold with isoperimetric profile I . Assume that all solutions of the isoperimetric problem in M are smooth. Then for each $\epsilon > 0$, the subset

$$X^{I,\epsilon} = \{B \in X^I \mid \epsilon \leq \text{vol}(B) \leq \text{vol}(M) - \epsilon\}$$

is compact in $C^{2,\alpha}$ topology.

Proof. The proof is essentially the same as in Lemma 8 of [4]. It depends on the fact that the $C^{2,\alpha}$ topology is equivalent to the flat topology of Geometric Measure Theory on the space of smooth isoperimetric domains. Compactness follows from the compactness theorem for currents of bounded mass and boundary mass, provided one assumes that flat limits of smooth minimizers (they are again minimizers) are smooth. \square

5. Proof of Theorem 1.1

Proof. By definition, elements of X^I coincide with solutions of the isoperimetric problem. We know from [6] that elements of X^I of small volume are nearly round balls emanating from the unique maximum of scalar curvature (see also [8]). They form a smooth connected 1-dimensional manifold. In other words, for ϵ small enough, X^I is the union of $X^{I,\epsilon}$ and of two smooth one-ended arcs,

$$X^{I,\leq\epsilon} = \{B \in X^I \mid \text{vol}(B) \leq \epsilon\} \quad \text{and} \quad X^{I,\geq V-\epsilon} = \{B \in X^I \mid \text{vol}(B) \geq \text{vol}(M) - \epsilon\}.$$

From Proposition 4.1, it follows that X^I has exactly two ends.

According to [4], I is semi-analytic on $[0, \text{vol}(M)]$. If smooth on $(0, \text{vol}(M))$, it must be analytic on the same interval. Consider the subset $D \subset X^I$ consisting of domains diffeomorphic to the n -ball. Since D is open and closed in X^I , it inherits the structure of a finite dimensional real analytic space. Let $D = D_N \supset D_{N-1} \supset \cdots \supset D_1 \supset D_0$ be the filtration provided by Proposition 1. The subset D_1 is a real analytic space of dimension 1. Let D'_1 denote the connected component of D_1 which contains $X^{I,\leq\epsilon}$.

By Corollary 4, D'_1 has a finite even number of ends. Since it contains one of the two ends of X^I , it must contain the other. In other words, there exists a domain B in M which is diffeomorphic to a n -ball, and whose complement is diffeomorphic to a n -ball as well. This implies that M is homeomorphic to a sphere. This proves Theorem 1.1. In fact, under the assumptions of Theorem 1.1, there is a global one parameter analytic family of domains inflating from the maximum point of curvature and ultimately engulfing the whole manifold while returning back to it. Indeed, D'_1 is a finite connected graph with two ends. Let A be a maximal element of the ordered set of chains of edges of D'_1 , containing $X^{I,\leq\epsilon}$, and which admit parametrizations by real analytic immersions. Proposition 3 asserts that such a maximal chain cannot end at a singular point, thus must contain $X^{I,\geq V-\epsilon}$ as well. Note that A need not be embedded. \square

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