Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 347 (2009) 45–48
Comptes Rendus Mathematique
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

Stability estimates on general scalar balance laws

Rinaldo M. Colombo, Magali Mercier [1], Massimiliano D. Rosini

Department of Mathematics, Brescia University, Via Branze 38, 25133 Brescia, Italy

Received 14 July 2008; accepted 29 October 2008

Available online 26 November 2008

Presented by Jean-Michel Bony

Abstract

Consider the general scalar balance law in N space dimensions $\partial_t u + \operatorname{Div} f(t,x,u) = F(t,x,u)$. Under suitable assumptions on f and F, we provide bounds on the total variation of the solution. Based on this first result, we establish estimates on the dependence of the solutions from f and F. In the more particular cases considered in the literature, the present estimate reduces to the known ones. ***To cite this article: R.M. Colombo et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Estimation de la variation totale et stabilité pour des lois de conservations scalaires généralisées. Nous considérons ici une loi de conservation généralisée en dimension N : $\partial_t u + \operatorname{Div} f(t,x,u) = F(t,x,u)$. Sous des hypothèses adaptées pour f et F, nous obtenons une borne de la variation totale de la solution. À partir de ce résultat, il est alors possible de donner une estimation de la dépendance des solutions au flot f et au terme source F. Dans les cas particuliers déjà étudiés, notre résultat se réduit à ceux déjà connus. ***Pour citer cet article : R.M. Colombo et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let $f \in \mathbf{C}^2(\overline{\mathbb{R}}_+ \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$, $F \in \mathbf{C}^1(\overline{\mathbb{R}}_+ \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$ and $u_o \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$. Then, the classical result [8, Theorem 5] ensures the well posedness of the Cauchy problem

$$\begin{cases} \partial_t u + \operatorname{Div} f(t,x,u) = F(t,x,u), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,x) = u_o(x), & x \in \mathbb{R}^N. \end{cases} \tag{1}$$

In the present Note, under suitable further assumptions on the flow f and on the source F, we state that the solution $u(t)$ to (1) is in $\mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ and provide bounds on its total variation.

This result allows us to obtain estimates on the dependence of the solution on f, F and u_o. Similar results were obtained in [2, Theorem 2.1] in the case of systems of conservation laws in 1 space dimension, with $f = f(u)$ and

E-mail addresses: rinaldo.colombo@unibs.it (R.M. Colombo), mercier@math.univ-lyon1.fr (M. Mercier), massimilianorosini@gmail.com (M.D. Rosini).

[1] Permanent address: Université de Lyon, Université Lyon 1, École Centrale de Lyon, INSA de Lyon, CNRS UMR 5208, Institut Camille Jordan, 43 boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.11.005

$F = 0$. In the case of a scalar equation with $f = f(u)$, $F = 0$ and with N space dimensions, the same problem was addressed by Bouchut and Perthame [3] who proved, among other results, the following estimate (that was already known, see [6,9]):

$$\big\|u(t) - v(t)\big\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \leqslant \|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} + C\,\mathrm{TV}(u_o)\,\mathbf{Lip}(f - g)t. \tag{2}$$

The estimate proved in Theorem 2.4 below reduces to (2) as soon as $f = f(u)$ and $F = 0$. In this context, we recall that [2, Theorem 2.6] provides a sharp estimate in the scalar 1D case with $f = f(u)$ and $F = 0$.

The case of x-dependent flows was considered in [4] and [7], where it is assumed that $f(x,u) = k(x)v(u)$. However, in both papers, the resulting estimate holds under the further assumption that the solutions have uniformly bounded total variation. Here, the bound on $\mathrm{TV}(u(t))$ is not assumed, but proved.

All proofs, together with an application to a radiating gas model, are deferred to [5].

2. Main results

Introduce the notation: $\overline{\mathbb{R}}_+ = [0, +\infty[$, $\mathbb{R}_+ =]0, +\infty[$, N is a positive integer and $\Omega = \overline{\mathbb{R}}_+ \times \mathbb{R}^N \times \mathbb{R}$. For a vector valued function $f = f(t,x,u)$ with $u = u(t,x)$, $\mathrm{Div}\, f$ stands for the total divergence while $\mathrm{div}\, f$, respectively ∇f, denotes the partial divergence, respectively gradient, with respect to the space variables. ∂_u and ∂_t are the usual partial derivatives. Thus, $\mathrm{Div}\, f = \mathrm{div}\, f + \partial_u f \cdot \nabla u$.

Recall the definition of weak entropy solution to (1), see [8, Definition 1]:

Definition 2.1. A bounded measurable function $u : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ is a weak entropy solution to (1) if:

1. for any constant $k \in \mathbb{R}$ and any test function $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \overline{\mathbb{R}}_+)$
$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^N} \big[(u-k)\partial_t\varphi + \big[f(t,x,u) - f(t,x,k)\big]\nabla\varphi + \big[F(t,x,u) - \mathrm{div}\, f(t,x,k)\big]\varphi\big]\,\mathrm{sign}(u-k)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0;$$
2. there exists a set $\mathcal{E}$ of zero measure in $\overline{\mathbb{R}}_+$ such that for $t \in \overline{\mathbb{R}}_+ \setminus \mathcal{E}$ the function $u(t,x)$ is defined almost everywhere in $\mathbb{R}^N$ and $\lim_{t\in\overline{\mathbb{R}}_+\setminus\mathcal{E},\, t\to 0} \int_{\|x\|\leqslant r} |u(t,x) - u_o(x)|\,\mathrm{d}x = 0$ for any $r > 0$.

We refer to [1] as general references for the theory of **BV** functions. In particular, recall the following basic definition, see [1, Definition 3.4 and Theorem 3.6]:

$$\mathrm{TV}(u) = \sup\left\{\int_{\mathbb{R}^N} u\,\mathrm{div}\,\psi\,\mathrm{d}x\colon\ \psi \in \mathbf{C}_c^1\big(\mathbb{R}^N;\mathbb{R}^N\big) \text{ and } \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \leqslant 1\right\},$$
$$\mathbf{BV}\big(\mathbb{R}^N;\mathbb{R}\big) = \big\{u \in \mathbf{L}^1_{\mathrm{loc}}\big(\mathbb{R}^N;\mathbb{R}\big)\colon\ \mathrm{TV}(u) < +\infty\big\}.$$

The following sets of assumptions will be of use below:

$$(\mathrm{H1}):\quad \begin{cases} f \in \mathbf{C}^2(\Omega;\mathbb{R}^N), \quad F \in \mathbf{C}^1(\Omega;\mathbb{R}), \\ \partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^N), \quad \partial_u(F - \mathrm{div}\, f) \in \mathbf{L}^\infty(\Omega;\mathbb{R}), \quad F - \mathrm{div}\, f \in \mathbf{L}^\infty(\Omega;\mathbb{R}), \end{cases}$$
$$(\mathrm{H2}):\quad \begin{cases} f \in \mathbf{C}^2(\Omega;\mathbb{R}^N), \quad F \in \mathbf{C}^1(\Omega;\mathbb{R}), \\ \partial_t\partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^N), \quad \partial_t\,\mathrm{div}\, f \in \mathbf{L}^\infty(\Omega;\mathbb{R}), \quad \partial_t F \in \mathbf{L}^\infty(\Omega;\mathbb{R}), \\ \nabla\partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N}), \quad \int_{\mathbb{R}_+}\int_{\mathbb{R}^N} \|\nabla(F - \mathrm{div}\, f)(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^N)}\,\mathrm{d}x\,\mathrm{d}t < +\infty, \end{cases}$$
$$(\mathrm{H3}):\quad \begin{cases} f \in \mathbf{C}^1(\Omega;\mathbb{R}^N), \quad F \in \mathbf{C}^0(\Omega;\mathbb{R}), \quad \partial_u F \in \mathbf{L}^\infty(\Omega;\mathbb{R}), \\ \partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^N), \quad \int_{\mathbb{R}_+}\int_{\mathbb{R}^N} \|(F - \mathrm{div}\, f)(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}t < +\infty. \end{cases}$$

Assumption (H1) is sufficient for the classical results by Kružkov [8, Theorem 1 and Theorem 5] to hold.

Theorem 2.2 *(Kružkov). Let* (H1) *hold. For any $u_o \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$, there exists a unique right continuous weak entropy solution u to* (1) *in $\mathbf{L}^\infty(\overline{\mathbb{R}}_+; \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}))$. Moreover, if a sequence $u_o^n \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$ converges to u_o in $\mathbf{L}^1_{\mathrm{loc}}$, then for all $t > 0$ the corresponding solutions $u^n(t)$ converge to $u(t)$ in $\mathbf{L}^1_{\mathrm{loc}}$.*

The next result contains the estimate on the total variation, a key point in the stability proof below:

Theorem 2.3. *Assume that* (H1) *and* (H2) *hold. Let* $u_o \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$. *Then, the weak entropy solution* u *of* (1) *satisfies* $u(t) \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ *for all* $t > 0$. *Moreover, let*

$$\kappa_o = N W_N \big((2N+1)\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}\big) \quad \text{and} \quad W_N = \int_0^{\pi/2} (\cos\theta)^N \,\mathrm{d}\theta. \tag{3}$$

Then,

$$\mathrm{TV}\big(u(T)\big) \leqslant \mathrm{TV}(u_o)\mathrm{e}^{\kappa_o T} + N W_N \int_0^T \mathrm{e}^{\kappa_o(T-t)} \int_{\mathbb{R}^N} \big\|\nabla(F - \operatorname{div} f)(t,x,\cdot)\big\|_{\mathbf{L}^\infty} \,\mathrm{d}x\,\mathrm{d}t. \tag{4}$$

This estimate is optimal in the following senses:

(i) If f is independent from x and $F = 0$, then $\kappa_o = 0$ and the integrand in the right hand side above vanishes. Hence, (4) reduces to the well known optimal bound $\mathrm{TV}(u(t)) \leqslant \mathrm{TV}(u_o)$.

(ii) In the 1D case, if f and F are both independent from t and u, then $\kappa_o = 0$ and (1) reduces to the ordinary differential equation $\partial_t u = F - \operatorname{div} f$. In this case, (4) becomes

$$\mathrm{TV}\big(u(t)\big) \leqslant \mathrm{TV}(u_o) + t\,\mathrm{TV}(F - \operatorname{div} f).$$

(iii) If $f = 0$ and $F = F(t)$ then, trivially, $\mathrm{TV}(u(t)) = \mathrm{TV}(u_o)$ and (4) is optimal.

Let now (f, F), (g, G) verify (H1) and $u_o, v_o \in \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$. We want to prove estimates for $u - v$ in terms of $f - g$, $F - G$ and $u_o - v_o$, u being the entropy solution of (1) and v being the entropy solution of

$$\begin{cases} \partial_t v + \operatorname{Div} g(t,x,v) = G(t,x,v), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ v(0,x) = v_o(x), & x \in \mathbb{R}^N. \end{cases}$$

Similar estimates were derived in [3] when f and g depend only on u. Here, we add the (t,x)-dependence.

Theorem 2.4. *Let* $(f,F), (g,G)$ *verify* (H1), (f,F) *verify* (H2) *and* $(f-g, F-G)$ *verify* (H3). *Let* $u_o, v_o \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$. *We denote* κ_o *and* W_N *as in* (3), $\kappa = 2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|\partial_u(F-G)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}$ *and* $M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)}$. *Then, for any* $T, R > 0$, $x_o \in \mathbb{R}^N$,

$$\begin{aligned} \int_{\|x-x_o\|\leqslant R} \big|u(T,x) - v(T,x)\big|\,\mathrm{d}x \leqslant{}& \mathrm{e}^{\kappa T} \int_{\|x-x_o\|\leqslant R+MT} \big|u_o(x) - v_o(x)\big|\,\mathrm{d}x + \frac{\mathrm{e}^{\kappa_o T} - \mathrm{e}^{\kappa T}}{\kappa_o - \kappa}\mathrm{TV}(u_o)\big\|\partial_u(f-g)\big\|_{\mathbf{L}^\infty} \\ &+ N W_N \int_0^T \frac{\mathrm{e}^{\kappa_o(T-t)} - \mathrm{e}^{\kappa(T-t)}}{\kappa_o - \kappa} \int_{\mathbb{R}^N} \big\|\nabla(F - \operatorname{div} f)(t,x,\cdot)\big\|_{\mathbf{L}^\infty} \,\mathrm{d}x\,\mathrm{d}t\,\big\|\partial_u(f-g)\big\|_{\mathbf{L}^\infty} \\ &+ \int_0^T \mathrm{e}^{\kappa(T-t)} \int_{\|x-x_o\|\leqslant R+M(T-t)} \big\|\big((F-G) - \operatorname{div}(f-g)\big)(t,x,\cdot)\big\|_{\mathbf{L}^\infty} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5}$$

Formally, the above inequality is undefined for $\kappa = \kappa_o$. However, as shown in [5], when $(\kappa - \kappa_o) \to 0$ the right hand side above has a finite limit which bounds the distance between solutions. Note that (4), as well as (5), does not depend on all second derivatives, hence the regularity requirements on f can be relaxed, see [5] for details. Besides, (5) is optimal in the cases considered before. Assume $u_o, v_o \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$.

(i) In the standard case of a conservation law, i.e. when $F = G = 0$ and f, g are independent of x, we have $\kappa_o = \kappa = 0$ and (5) becomes, see [2, Theorem 2.1],

$$\left\|u(T) - v(T)\right\|_{\mathbf{L}^1} \leqslant \|u_o - v_o\|_{\mathbf{L}^1} + T\,\mathrm{TV}(u_o)\left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty}.$$

(ii) If $\partial_u f = \partial_u g = 0$ and $\partial_u F = \partial_u G = 0$, then $\kappa_o = \kappa = 0$ and (5) now reads

$$\left\|u(T) - v(T)\right\|_{\mathbf{L}^1} \leqslant \|u_o - v_o\|_{\mathbf{L}^1} + \int_0^T \left\|\big((F-G) - \mathrm{div}(f-g)\big)(t)\right\|_{\mathbf{L}^1} \mathrm{d}t.$$

(iii) If (f, F) and (g, G) are dependent only on x, then (5) reduces to

$$\left\|u(T) - v(T)\right\|_{\mathbf{L}^1} \leqslant \|u_o - v_o\|_{\mathbf{L}^1} + T\left\|(F-G) - \mathrm{div}(f-g)\right\|_{\mathbf{L}^1}.$$

References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
[2] S. Bianchini, R.M. Colombo, On the stability of the standard Riemann semigroup, Proc. Amer. Math. Soc. 130 (7) (2002) 1961–1973 (electronic).
[3] F. Bouchut, B. Perthame, Kružkov's estimates for scalar conservation laws revisited, Trans. Amer. Math. Soc. 350 (7) (1998) 2847–2870.
[4] G.-Q. Chen, K.H. Karlsen, Quasilinear anisotropic degenerate parabolic equations with time-space dependent diffusion coefficients, Comm. Pure Appl. Anal. 4 (2) (2005) 241–266.
[5] R.M. Colombo, M. Mercier, M.D. Rosini, Stability and total variation estimates on general scalar balance laws, Comm. Math. Sci., submitted for publication.
[6] C.M. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, J. Math. Anal. Appl. 38 (1972) 33–41.
[7] K.H. Karlsen, N.H. Risebro, On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients, Discrete Contin. Dynam. Syst. 9 (5) (2003) 1081–1104.
[8] S.N. Kružkov, First order quasilinear equations with several independent variables, Mat. Sb. 81 (123) (1970) 228–255.
[9] B.J. Lucier, A moving mesh numerical method for hyperbolic conservation laws, Math. Comp. 46 (173) (1986) 59–69.