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Complex Analysis

A solution of Gromov's Vaserstein problem

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Abstract

We announce that a null-homotopic holomorphic mapping from a finite dimensional reduced Stein space into $\mathrm{SL}_n(\mathbb{C})$ can be factored into a finite product of unipotent matrices with holomorphic entries. ***To cite this article: B. Ivarsson, F. Kutzschebauch, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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Résumé

Une solution du problème de Vaserstein tel qu'énoncé par Gromov. Nous annonçons qu'une application holomorphe homotopiquement triviale d'un espace de Stein réduit de dimension finie vers $\mathrm{SL}_n(\mathbb{C})$ peut être factorisée par un produit fini de matrices unipotentes à coefficients holomorphes. ***Pour citer cet article : B. Ivarsson, F. Kutzschebauch, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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Version française abrégée

Dans cette Note, nous présentons une solution affirmative complète au problème de Vaserstein tel que posé par Gromov

Problème de Vaserstein. ([14, sec 3.5.G].) Est-ce que toute application holomorphe $\mathbb{C}^n \to \mathrm{SL}_m(\mathbb{C})$ se décompose en un produit fini d'applications holomorphes qui envoient $\mathbb{C}^n$ vers des sous-groupes unipotents de $\mathrm{SL}_m(\mathbb{C})$?

Nous énonçons ainsi

Théorème 0.1. *Soit X un espace réduit de Stein de dimension finie et $f : X \to \mathrm{SL}_m(\mathbb{C})$ une application holomorphe homotopiquement triviale. Alors il existe un entier naturel K et des applications holomorphes $G_1, \ldots, G_K : X \to \mathbb{C}^{m(m-1)/2}$ tels que*

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$$f(x)=\begin{pmatrix}1 & 0\\ G_1(x) & 1\end{pmatrix}\begin{pmatrix}1 & G_2(x)\\ 0 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & G_K(x)\\ 0 & 1\end{pmatrix}$$

est un produit de matrices unipotentes triangulaires supérieures et inférieures.

1. Introduction

It is standard material in a Linear Algebra course that the group $\mathrm{SL}_m(\mathbb{C})$ is generated by elementary matrices $E_m+\alpha e_{ij}$, $i\neq j$, i.e., matrices with 1's on the diagonal and all entries outside the diagonal are zero, except one entry. Equivalently every matrix $A\in\mathrm{SL}_m(\mathbb{C})$ can be written as a finite product of upper and lower diagonal unipotent matrices (in interchanging order). The same question for matrices in $\mathrm{SL}_m(R)$ where R is a commutative ring instead of the field $\mathbb{C}$ is much more delicate. For example if R is the ring of complex valued functions (continuous, smooth, algebraic or holomorphic) from a space X the problem amounts to find for a given map $f:X\to\mathrm{SL}_m(\mathbb{C})$ a factorization as a product of upper and lower diagonal unipotent matrices

$$f(x)=\begin{pmatrix}1 & 0\\ G_1(x) & 1\end{pmatrix}\begin{pmatrix}1 & G_2(x)\\ 0 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & G_N(x)\\ 0 & 1\end{pmatrix}$$

where the G_i are maps $G_i:X\to\mathbb{C}^{m(m-1)/2}$.

Since any product of (upper and lower diagonal) unipotent matrices is homotopic to a constant map (multiplying each entry outside the diagonals by $t\in[0,1]$ we get a homotopy to the identity matrix), one has to assume that the given map $f:X\to\mathrm{SL}_m(\mathbb{C})$ is homotopic to a constant map or as we will say null-homotopic. In particular this assumption holds if the space X is contractible.

This very general problem has been studied in the case of polynomials of n variables. For $n=1$, i.e., $f:X\to\mathrm{SL}_m(\mathbb{C})$ a polynomial map (the ring R equals $\mathbb{C}[z]$) it is an easy consequence of the fact that $\mathbb{C}[z]$ is an Euclidean ring that such f factors through a product of upper and lower diagonal unipotent matrices. For $m=n=2$ the following counterexample was found by Cohn [1]: the matrix

$$\begin{pmatrix}1-z_1z_2 & z_1^2\\ -z_2^2 & 1+z_1z_2\end{pmatrix}\in\mathrm{SL}_2\big(\mathbb{C}[z_1,z_2]\big)$$

does not decompose as a finite product of unipotent matrices.

For $m\geqslant 3$ (and any n) it is a deep result of Suslin [17] that any matrix in $\mathrm{SL}_m(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices. More results in the algebraic setting can be found in [17] and [10]. For a connection to the Jacobian problem on $\mathbb{C}^2$ see [20].

In the case of continuous complex valued functions on a topological space X the problem was studied and partially solved by Thurston and Vaserstein [18] and then finally solved by Vaserstein [19, Theorem 4].

It is natural to consider the problem for rings of holomorphic functions on Stein spaces, in particular on $\mathbb{C}^n$. Explicitly this problem was posed by Gromov in his groundbreaking paper [14] where he extends the classical Oka–Grauert theorem from bundles with homogeneous fibers to fibrations with elliptic fibers, e.g., fibrations admitting a dominating spray. In spite of the above mentioned result of Vaserstein he calls it the

Vaserstein problem. (See [14, sec 3.5.G].) Does every holomorphic map $\mathbb{C}^n\to\mathrm{SL}_m(\mathbb{C})$ decompose into a finite product of holomorphic maps sending $\mathbb{C}^n$ into unipotent subgroups in $\mathrm{SL}_m(\mathbb{C})$?

In this Note we announce a complete positive solution of Gromov's Vaserstein problem, namely

Theorem 1. *Let X be a finite dimensional reduced Stein space and $f:X\to\mathrm{SL}_m(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number K and holomorphic mappings $G_1,\ldots,G_K:X\to\mathbb{C}^{m(m-1)/2}$ such that*

$$f(x)=\begin{pmatrix}1 & 0\\ G_1(x) & 1\end{pmatrix}\begin{pmatrix}1 & G_2(x)\\ 0 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & G_N(x)\\ 0 & 1\end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices.

We have the following corollary which in particular solves Gromov's Vaserstein problem:

Corollary 1.1. *Let X be a finite dimensional reduced Stein space that is topologically contractible and $f : X \to \mathrm{SL}_m(\mathbb{C})$ be a holomorphic mapping. Then there exist a natural number K and holomorphic mappings $G_1, \dots, G_K : X \to \mathbb{C}^{m(m-1)/2}$ such that*

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices.

By the definition of the Whitehead K_1-group of a ring, see [15, page 61], this implies:

Corollary 1.2. *Let X be a finite dimensional reduced Stein space that is topologically contractible and denote by $\mathcal{O}(X)$ the ring of holomorphic functions on X. Then $SK_1(\mathcal{O}(X))$ is trivial and the determinant induces an isomorphism* $\det : K_1(\mathcal{O}(X)) \to \mathcal{O}(X)^\star$.

The method of proof is an application of the Oka–Grauert–Gromov-principle to certain stratified fibrations. The existence of a topological section for these fibrations we deduce from Vaserstein's result.

We need the principle in it's strongest form suggested by Gromov, completely proven by Forstnerič and Prezelj [4], see also Forstnerič [3, Theorem 8.3]. After the Gromov–Eliashberg embedding theorem for Stein manifolds (see [2,16]) this is to our knowledge the second time this holomorphic h-principle has an application which goes beyond the classical results of Grauert, Forster and Rammspott [13,12,11,5,9,8,7,6].

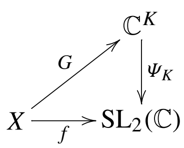
2. Sketch of the proof for $\mathrm{SL}_2(\mathbb{C})$

All complex spaces considered in this paper will be assumed reduced and we will not repeat this every time. We call a complex space X finite dimensional if its smooth part $X \setminus X^{\mathrm{sing}}$ has finite dimension. Note that this does not imply that they have finite embedding dimension.

Define $\Psi_K : \mathbb{C}^K \to \mathrm{SL}_2(\mathbb{C})$ as

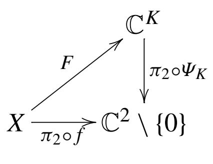
$$\Psi_K(z_1, \dots, z_K) = \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & z_K \\ 0 & 1 \end{pmatrix}.$$

We want to show the existence of a holomorphic map $G = (G_1, \dots, G_K) : X \to \mathbb{C}^K$ such that



is commutative. The result by Vaserstein shows the existence of a continuous map such that the diagram above is commutative.

We prove the result applying the Oka–Grauert–Gromov principle for sections of holomorphic submersions over X coming from the diagram



where we define the projection $\pi_2 : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}^2 \setminus \{0\}$ to be the projection of a matrix in $\mathrm{SL}_2(\mathbb{C})$ to its second row.

However, the map $\Phi_K = \pi_2 \circ \Psi_K : \mathbb{C}^K \to \mathbb{C}^2 \setminus \{0\}$ is not submersive everywhere. We have the following result which describes the situation:

Lemma 2.1. *The mapping $\Phi_K = \pi_2 \circ \Psi_K : \mathbb{C}^K \to \mathbb{C}^2 \setminus \{0\}$ is a holomorphic submersion exactly at points $\mathbb{C}^K \setminus S_K$, where for $K \geqslant 2$,*

$$S_K = \{(z_1, \ldots, z_K) \in \mathbb{C}^K \colon z_1 = \cdots = z_{K-1} = 0\}$$

and the submersion $\Phi_K = \pi_2 \circ \Psi_K : \mathbb{C}^K \setminus S_K \to \mathbb{C}^2 \setminus \{0\}$ is surjective when $K \geqslant 3$.

The following is crucial for the proof:

Lemma 2.2. *The holomorphic submersions $\Phi_K : \mathbb{C}^K \setminus S_K \to \mathbb{C}^2 \setminus \{0\}$, for $K \geqslant 3$, admit stratified sprays* (*for a definition of sprays and stratified sprays see* [14] *and* [4]).

Proof. Write $\Phi_K(z_1, \ldots, z_K) = (P_K(z_1, \ldots, z_K), Q_K(z_1, \ldots, z_K))$ and note that

$$P_K(z_1, \ldots, z_K) = P_{K-1}(z_1, \ldots, z_{K-1})$$

and

$$Q_K(z_1, \ldots, z_K) = Q_{K-1}(z_1, \ldots, z_{K-1}) + z_K P_{K-1}(z_1, \ldots, z_{K-1})$$

when K is even and similarly $P_K = z_K P_{K-1} + Q_{K-1}$ and $Q_K = Q_{K-1}$ when K is odd. We concentrate on the case when K is even. The odd case is handled in the same way. Let $(a, b) \in \mathbb{C}^2 \setminus \{0\}$ and study the fiber $P_K = a$, $Q_K = b$. When $a \neq 0$ the fiber is a graph in $\mathbb{C}^{K-1} \times \mathbb{C}_{z_K}$ over $P_{K-1} = a$ in $\mathbb{C}^{K-1}$ since $z_K = (b - Q_{K-1})/a$. When $a = 0$ the fiber is $\Phi_{K-1}^{-1}(0, b) \times \mathbb{C}_{z_K}$ and, since $b \neq 0$ in this case, $\Phi_{K-1}^{-1}(0, b)$ is a graph in $\mathbb{C}^{K-2} \times \mathbb{C}_{z_{K-1}}$ over $Q_{K-2} = b$ in $\mathbb{C}^{K-2}$. So in both cases the fibers described by two polynomial equations are reduced to graphs over a surface described by a single polynomial equation. A standard way of producing sprays is to use flows of globally integrable tangential holomorphic vector fields spanning the tangent space at each point of the fiber and this is the method we will use. We will use $\mathbb{C}^2 \setminus \{0\} \supset \{(a, b) \in \mathbb{C}^2 \setminus \{0\}; a = 0\} \supset \emptyset$ to stratify $\mathbb{C}^2 \setminus \{0\}$ and only describe the construction of the spray in the stratum $\{(a, b) \in \mathbb{C}^2 \setminus \{0\}; a \neq 0\}$. The stratum $\{(a, b) \in \mathbb{C}^2 \setminus \{0\}; a = 0\}$ is handled similarly. We need to find globally integrable tangential holomorphic vector fields that span the tangent space of $P_{K-1} = a \neq 0$ at each point of the surface. We point out that non-smooth points of $P_{K-1} = a$ are contained in S_K if they exist. We claim that the vector fields

$$V_{ij} = \frac{\partial P_{K-1}}{\partial z_i} \frac{\partial}{\partial z_j} - \frac{\partial P_{K-1}}{\partial z_j} \frac{\partial}{\partial z_i}, \quad 1 \leqslant i < j \leqslant K$$

have the properties needed. They obviously are tangential and one easily check that they span the tangent space of the fiber at each point. In order to realize that the vector fields are globally integrable note that P_{K-1} is a polynomial that is no more than linear in each variable separably. Also since $\partial P_{K-1}/\partial z_i$ is independent of z_i and $\partial P_{K-1}/\partial z_j$ is independent of z_j the flow of V_{ij} is the solution of a system of two independent differential equations that are both globally integrable. This concludes the sketch of the proof of the lemma. □

Lemma 2.2 allows us to prove the following proposition:

Proposition 2.3. *Let X be a finite dimensional reduced Stein space and $f : X \to \mathrm{SL}_2(\mathbb{C})$ be a null-homotopic holomorphic map. Assume that there exists a natural number K and a continuous map $F : X \to \mathbb{C}^K \setminus S_K$ such that*

$$\begin{array}{ccc} & & \mathbb{C}^K \setminus S_K \\ & \overset{F}{\nearrow} & \downarrow{\scriptstyle \pi_2 \circ \Psi_K} \\ X & \xrightarrow[\pi_2 \circ f]{} & \mathbb{C}^2 \setminus \{0\} \end{array}$$

is commutative. Then there exists a holomorphic map $G : X \to \mathbb{C}^K \setminus S_K$ such that the diagram is commutative.

Proof. Put $Y = \mathbb{C}^K \setminus S_K$ and $p = \pi_2 \circ f$. Define the pull-back of $(Y, \Phi_K, \mathbb{C}^2 \setminus \{0\})$ via $p : X \to \mathbb{C}^2 \setminus \{0\}$ as $(p^\star Y, p^\star \Phi_K, X)$ where

$$p^\star Y = \{(x, Z) \in X \times Y; p(x) = \Phi_K(Z)\}$$

and $p^\star\Phi_K(x,Z)=x$. Using that Φ_K is a holomorphic submersion we see that $p^\star\Phi_K$ is a holomorphic submersion. The continuous mapping F defines a continuous section

$$p^\star F(x)=\big(x,F(x)\big)$$

of $(p^\star Y, p^\star\Phi_K, X)$. From Lemma 2.2 and the finite dimensionality of X it follows that $(p^\star Y, p^\star\Phi_K, X)$ admits stratified sprays. Now the result follows by the Oka–Grauert–Gromov-principle (see [14] and [4]). □

Proof. (Proof of Theorem 1 for $SL_2(\mathbb{C})$.) By a result of Vaserstein [19, Theorem 4] we have a continuous map $F: X\to\mathbb{C}^{K'}$ for some natural number K' such that $f(x)=\Psi_{K'}(F(x))$. Using Lemma 2.1 we see that $F=(F_1,\dots,F_{K'},1,0,-1)$ gives a map from X into $\mathbb{C}^{K'+3}\setminus S_{K'+3}$ and putting $K=K'+3$ we have $f(x)=\Psi_K(F(x))$. It follows that $\Psi_K(F(x))f(x)^{-1}=E_2$. Using Proposition 2.3 we know that there is holomorphic map G such that

$$\Phi_K\big(F(x)\big)=\pi_2\big(f(x)\big)=\Phi_K\big(G(x)\big)$$

that is the last rows of the matrices $\Psi_K(F(x))$ and $\Psi_K(G(x))$ are equal.

Therefore

$$\Psi_K\big(G(x)\big)f(x)^{-1}=\begin{pmatrix} a(x) & b(x)\\ 0 & 1\end{pmatrix}$$

where a and b are holomorphic and moreover $a\equiv 1$ since the left hand side is in $SL_2(\mathbb{C})$. Thus

$$f(x)=\Psi_K\big(G(x)\big)\begin{pmatrix} 1 & -b(x)\\ 0 & 1\end{pmatrix}$$

which solves the problem. □

The proof in the general case is by induction on the size of the matrices. The difficult part is to show Lemma 2.2. We are able to reduce strata-wise the m polynomial equations which define the fibers of Φ_K to one single polynomial equation. The special form of the occurring polynomials allows us to find a spray by constructing finitely many globally integrable holomorphic vector fields which span the tangent space of the fibers at each point. The details will appear elsewhere.

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