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Partial Differential Equations

New characterization of the kernel of the n-dimensional Laplace operator in exterior domains

Chérif Amrouche, Huy Hoang Nguyen

Laboratoire de mathématiques appliquées, CNRS UMR 5142, Université de Pau et des Pays de l'Adour, IPRA, avenue de l'Université, 64013 Pau, France

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Abstract

In this Note, we study the characterization of the kernel of the Laplace operator with Dirichlet boundary conditions in exterior domains. We consider data in weighted Sobolev spaces. ***To cite this article: C. Amrouche, Huy Hoang Nguyen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Nouvelle caractérisation du noyau du laplacien en domaine extérieur. Nous étudions dans cette Note la caractérisation du noyau de l'opérateur laplacien avec des conditions de Dirichlet au bord dans un ouvert extérieur. Nous considérons des données dans des espaces de Sobolev avec poids. ***Pour citer cet article : C. Amrouche, Huy Hoang Nguyen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Let Ω' be a bounded open region of $\mathbb{R}^n$ ($n \geqslant 2$), not necessarily connected, with a Lipschitz-continuous boundary Γ and let Ω be the complement of $\overline{\Omega'}$. We suppose that Ω' has a finite number of connected components and each connected component has a connected boundary, so that Ω is connected. For convenience, the origin of the coordinate frame is attached to $\overline{\Omega'}$. The purpose of this Note is to characterize the kernel $\mathcal{A}^{p,q}(\Omega)$ of the Laplace operator with Dirichlet boundary conditions:

$$\mathcal{A}^{p,q}(\Omega) = \left\{ z \in W_0^{1,p}(\Omega) + W_0^{1,q}(\Omega);\ \Delta z = 0 \text{ in } \Omega \text{ and } z = 0 \text{ on } \Gamma \right\}. \tag{1}$$

The motivation for studying the space $\mathcal{A}^{p,q}(\Omega)$ is the regularity problem of Laplace equation. Let $f \in W_0^{-1,p}(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $u \in W_0^{1,p}(\Omega)$ be a solution of the following system:

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = g \quad \text{on } \Gamma.$$

E-mail addresses: cherif.amrouche@univ-pau.fr (C. Amrouche), huy-hoang.nguyen@etud.univ-pau.fr (Huy Hoang Nguyen).

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Recall that a solution u exists and is unique if and only if f and g satisfy the compatibility condition: for any $\varphi \in \mathcal{A}^{p'}(\Omega)$,

$$\langle f, \varphi\rangle_{W_0^{-1,p}(\Omega)\times \mathring{W}_0^{1,p'}(\Omega)} = \left\langle g, \frac{\partial \varphi}{\partial \boldsymbol{n}}\right\rangle_{W^{1-\frac{1}{p},p}(\Gamma)\times W^{\frac{-1}{p},p'}(\Gamma)}. \tag{2}$$

If, in addition, $f \in W_0^{-1,q}(\Omega)$, $g \in W^{1-\frac{1}{q},q}(\Gamma)$ with $p < q$ satisfying the compatibility condition (2) by replacing p by q, the question "Does the solution u belong to $W_0^{1,q}(\Omega)$?" arises. Since there exists $v \in W_0^{1,q}(\Omega)$ satisfying $-\Delta v = f$ in Ω and $v = g$ on Γ, from Theorem 2.1 we obtain $u - v \in \mathcal{A}^{p,q}(\Omega)$. Therefore, if $q < n$ or $q = n = 2$, then $u = v$ and $u \in W_0^{1,q}(\Omega)$. Otherwise, $u = v + \lambda \in W_0^{1,q}(\Omega)$ with $\lambda \in \mathcal{A}^{p,q}(\Omega)$.

Since the problem is posed in a n-dimensional exterior domain, it is important to specify the behavior at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces which provide a correct functional setting for unbounded domains (see [2] for more details). It means that the growth and decay of functions at infinity are expressed by means of weights, in particular, the function in these weighted Sobolev spaces satisfies an optimal weighted Poincaré-type inequality. In the whole text, bold characters are used for vector or matrix fields. We now introduce the definition of weighted Sobolev spaces and some its properties. A typical point in $\mathbb{R}^n$ is denoted by $\boldsymbol{x} = (x_1, \ldots, x_n)$ and its norm is given by $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. We define the weight function $\rho(\boldsymbol{x}) = 1 + r$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. We now define the weighted Sobolev space $W_0^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{w} \in L^p(\Omega),\ \nabla u \in \mathbf{L}^p(\Omega)\}$, where

$$w = \begin{cases} (1+r) & \text{if } p \neq n, \\ (1+r)\ln(2+r) & \text{if } p = n. \end{cases}$$

This space is a reflexive Banach space when endowed with the norm: $\|u\|_{W_0^{1,p}(\Omega)} = (\|\frac{u}{w}\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{\mathbf{L}^p(\Omega)})^{1/p}$. We note that the logarithmic weight only appears if $p = n$ and all the local properties of $W_0^{1,p}(\Omega)$ coincide with those of the corresponding classical Sobolev space $W^{1,p}(\Omega)$. We set $\mathring{W}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W_0^{1,p}(\Omega)}$ and we denote the dual space of $\mathring{W}_0^{1,p}(\Omega)$ by $W_0^{-1,p'}(\Omega)$, which is a space of distributions. When $\Omega = \mathbb{R}^n$, we have $W_0^{1,p}(\mathbb{R}^n) = \mathring{W}_0^{1,p}(\mathbb{R}^n)$. We have the algebraic and topological embeddings $W_0^{1,p}(\Omega) \hookrightarrow W_{-1}^{0,p}(\Omega)$ if $p \neq n$, where $W_{-1}^{0,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{1+r} \in L^p(\Omega)\}$. For all $\lambda \in \mathbb{N}^n$ where $0 \leqslant |\lambda| \leqslant 2$, the mapping $u \in W_0^{1,p}(\Omega) \to \partial^\lambda u \in W_0^{1-|\lambda|,p}(\Omega)$ is continuous. Also recall the following Sobolev embeddings (see [1]): $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ where $p^* = \frac{np}{n-p}$ and $1 < p < n$. Note that $\mathbb{R} \subset W_0^{1,p}(\Omega)$ if and only if $p \geqslant n$. We next set $\mathcal{A}^p(\Omega) = \{y \in W_0^{1,p}(\Omega);\ \Delta y = 0 \text{ in } \Omega \text{ and } y = 0 \text{ on } \Gamma\}$. In the two-dimensional space, let $U = \frac{1}{2\pi}\ln r$ be the fundamental solution of Laplace's equation. We now define

$$u_0 = U * \left(\frac{1}{|\Gamma|}\delta_\Gamma\right), \tag{3}$$

where δ_Γ is the distribution defined by $\forall \varphi \in \mathcal{D}(\mathbb{R}^2)$, $\langle \delta_\Gamma, \varphi\rangle = \int_\Gamma \varphi \,\mathrm{d}\sigma$.

The next lemma characterizes the kernel $\mathcal{A}^p(\Omega)$ (see [3]).

Lemma 1.1. *Let $1 < p < \infty$ and suppose that Γ is of class $C^{1,1}$.*

(i) *If $p < n$ or if $p = n = 2$, then $\mathcal{A}^p(\Omega) = \{0\}$.*

(ii) *If $p \geqslant n \geqslant 3$, then $\mathcal{A}^p(\Omega) = \{c(\lambda - 1);\ c \in \mathbb{R}\}$, where $\lambda \in \bigcap_{r > \frac{n}{n-1}} W_0^{1,r}(\Omega)$ is the unique solution of the following problem*

$$\Delta\lambda = 0 \quad \text{in } \Omega \quad \text{and} \quad \lambda = 1 \quad \text{on } \Gamma. \tag{4}$$

(iii) *If $p > n = 2$, then $\mathcal{A}^p(\Omega) = \{c(\mu - u_0);\ c \in \mathbb{R}\}$, where u_0 is defined by (3) and μ is the only solution in $\bigcap_{r>2} W_0^{1,r}(\Omega)$ of the problem*

$$\Delta\mu = 0 \quad \text{in } \Omega \quad \text{and} \quad \mu = u_0 \quad \text{on } \Gamma. \tag{5}$$

Remark 1.2. When Γ is the unit sphere of $\mathbb{R}^n$ ($n \geqslant 3$), then $\lambda = \frac{1}{|\boldsymbol{x}|^{n-2}}$. Note that $\nabla\lambda \in \mathbf{L}^{n/(n-1),\infty}(\mathbb{R}^n)$ and $\frac{\lambda}{w} \in L^{n/(n-1),\infty}(\mathbb{R}^n)$, where the weak-type space $L^{p,\infty}(\mathbb{R}^n)$ is defined as follows

$$u \in L^{p,\infty}(\mathbb{R}^n) \Leftrightarrow \sup_{t>0} t\Bigg(\int\limits_{\{\boldsymbol{x}\in\mathbb{R}^n,\, u(\boldsymbol{x})|>t\}} \mathrm{d}\boldsymbol{x}\Bigg)^{1/p} < \infty.$$

Then we will write $\lambda \in W^{1,n/(n-1)}_{0,\infty}(\mathbb{R}^n)$.

2. Main results

In this section, we give a theorem that characterizes the kernel $\mathcal{A}^{p,q}(\Omega)$ of the Laplace operator with Dirichlet boundary conditions: $\mathcal{A}^{p,q}(\Omega) = \{z \in W^{1,p}_0(\Omega) + W^{1,q}_0(\Omega);\ \Delta z = 0 \text{ in } \Omega \text{ and } z = 0 \text{ on } \Gamma\}$, with $1 < p < q < \infty$.

Theorem 2.1. *Let $1 < p < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$ boundary.*

(i) *If $q < n$ or if $q = n = 2$, then $\mathcal{A}^{p,q}(\Omega) = \{0\}$.*
(ii) *If $q \geqslant n \geqslant 3$, then $\mathcal{A}^{p,q}(\Omega) = \{c(\lambda - 1);\ c \in \mathbb{R}\}$ where $\lambda \in \bigcap_{r>\frac{n}{n-1}} W^{1,r}_0(\Omega)$ is the unique solution of the problem* (4).
(iii) *If $q > n = 2$, then $\mathcal{A}^{p,q}(\Omega) = \{c(\mu - u_0);\ c \in \mathbb{R}\}$ where $\mu \in \bigcap_{r>2} W^{1,r}_0(\Omega)$ is the unique solution of the problem* (5).

Proof. Let $z \in \mathcal{A}^{p,q}(\Omega)$, then $z = u - v$ with $u \in W^{1,p}_0(\Omega)$, $v \in W^{1,q}_0(\Omega)$ and $u = v$ on Γ. Let now $\tilde{v} \in W^{1,q}_0(\mathbb{R}^n)$ be an extension of v outside Ω. We set $\tilde{u} = u$ in Ω, $\tilde{u} = \tilde{v}$ outside Ω and $\tilde{z} = \tilde{u} - \tilde{v}$. It is easy to see that $\tilde{z}$ is in $W^{1,p}_0(\mathbb{R}^n) + W^{1,q}_0(\mathbb{R}^n)$ and $\tilde{z} = 0$ outside Ω. Set now $h = \Delta\tilde{z}$. As $\operatorname{supp} h \subset \Gamma$, then $h \in W^{-1,p}_0(\mathbb{R}^n)$.
A. *If* $n \geqslant 3$: We consider 3 following cases:
1) *The case* $\frac{n}{n-1} < p$: We know that there exists $w \in W^{1,p}_0(\mathbb{R}^n)$ such that $\Delta w = h$ in $\mathbb{R}^n$. The difference $w - \tilde{z}$ belongs to $W^{1,p}_0(\mathbb{R}^n) + W^{1,q}_0(\mathbb{R}^n)$ and is harmonic in $\mathbb{R}^n$. We begin by supposing that $q < n$. We deduce that $w = \tilde{z}$ in $\mathbb{R}^n$ and then w vanishes on Γ. Since $p < n$, thanks to Lemma 2.10 [3], w is unique and $w = 0$ in Ω, i.e., $z = 0$ in Ω. Now if $q \geqslant n$, there exists a constant c such that $w - \tilde{z} = c$ and $w = c$ on Γ. If $p < n$, from Lemma 2.10 [3], then w is unique and $w = c\lambda$ in Ω where $\lambda \in \bigcap_{r>\frac{n}{n-1}} W^{1,r}_0(\Omega)$ is the unique solution of the system (4). Therefore, we can deduce $z = c(\lambda - 1)$ in Ω. If $p \geqslant n$, it is easy to deduce that w is unique up to a constant and we still obtain that $z = c(\lambda - 1)$ in Ω.
2) *The case* $1 < p < \frac{n}{n-1}$: In the n-dimensional case, let $E(\boldsymbol{x}) = c_n|\boldsymbol{x}|^{2-n}$ be the fundamental solution of Laplace's equation. As $\delta \in W^{-1,p}_0(\mathbb{R}^n)$ is the Dirac distribution, then there exists a unique $w_0 \in W^{1,p}_0(\mathbb{R}^n)$ such that $\Delta w_0 = h - \delta\langle h, 1\rangle_{W^{-1,p}_0(\mathbb{R}^n)\times W^{1,p'}_0(\mathbb{R}^n)}$ in $\mathbb{R}^n$. We now set that $w = w_0 - E\langle h, 1\rangle_{W^{-1,p}_0(\mathbb{R}^n)\times W^{1,p'}_0(\mathbb{R}^n)}$. Then $\Delta w = h$ in $\mathbb{R}^n$ and $w - \tilde{z}$ is harmonic. The restriction of w to Ω belongs to $W^{1,p}_0(\Omega) + W^{1,r}_0(\Omega)$ for all $r > \frac{n}{n-1}$. The function w belongs to $W^{1,p}_0(\mathbb{R}^n) + W^{1,n/(n-1)}_{0,\infty}(\mathbb{R}^n)$, i.e., $\nabla w \in \mathbf{L}^p(\mathbb{R}^n) + \mathbf{L}^{n/(n-1),\infty}(\mathbb{R}^n)$. Hence, the difference $w - \tilde{z}$ belongs to $W^{1,p}_0(\mathbb{R}^n) + W^{1,n/(n-1)}_{0,\infty}(\mathbb{R}^n) + W^{1,q}_0(\mathbb{R}^n)$.
a) *The case* $q < n$: We deduce $w = \tilde{z}$ in $\mathbb{R}^n$ and $w = 0$ on Γ. Then $\Delta w_0 = 0$ in Ω and $w_0 = \langle h, 1\rangle E$ on Γ. As $p' > n$, for any $\varphi \in \mathcal{A}^{p'}(\Omega)$ and for any $\psi \in \mathcal{D}(\overline{\Omega})$, we have the following Green's formula

$$\int\limits_\Omega \psi\Delta\varphi\,\mathrm{d}\boldsymbol{x} = \int\limits_\Omega \varphi\Delta\psi\,\mathrm{d}\boldsymbol{x} + \left\langle\frac{\partial\varphi}{\partial\boldsymbol{n}}, \psi\right\rangle_\Gamma - \left\langle\varphi, \frac{\partial\psi}{\partial\boldsymbol{n}}\right\rangle_\Gamma,$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality between $W^{\frac{-1}{p},p'}(\Gamma)$ and $W^{1-\frac{1}{p},p}(\Gamma)$. Then, we deduce that $\int_\Omega \varphi\Delta\psi\,\mathrm{d}\boldsymbol{x} = -\langle\frac{\partial\varphi}{\partial\boldsymbol{n}}, \psi\rangle_\Gamma$. Thanks to the density of $\mathcal{D}(\overline{\Omega})$ in $W^{1,p}_0(\Omega)$, for all $\varphi \in \mathcal{A}^{p'}(\Omega)$ and for all $v \in W^{1,p}_0(\Omega)$, we have

$$\langle\Delta v, \varphi\rangle_{W^{-1,p}_0(\Omega)\times\mathring{W}^{1,p'}_0(\Omega)} = -\left\langle\frac{\partial\varphi}{\partial\boldsymbol{n}}, v\right\rangle_{W^{\frac{-1}{p},p'}(\Gamma)\times W^{1-\frac{1}{p},p}(\Gamma)}. \tag{6}$$

Applying (6) with $v = w_0 \in W_0^{1,p}(\Omega)$ and $\varphi = \lambda - 1 \in \mathcal{A}^{p'}(\Omega)$, we obtain $\langle h, 1\rangle \langle E, \frac{\partial\lambda}{\partial \boldsymbol{n}}\rangle_{W^{1-\frac{1}{p},p}(\Gamma)\times W^{\frac{-1}{p},p'}(\Gamma)} = 0$. Note that, $\langle E, \frac{\partial\lambda}{\partial \boldsymbol{n}}\rangle_\Gamma = \langle \frac{\partial E}{\partial \boldsymbol{n}}, \lambda\rangle_\Gamma = \int_\Gamma \frac{\partial E}{\partial \boldsymbol{n}}$. Let B_R the open ball of radius $R > 0$ centered at the origin such that $\Omega' \subset B_R$ and set that $\Omega_R = \Omega \cap B_R$. Then we have $0 = \int_{\Omega_R} \Delta E = \int_\Gamma \frac{\partial E}{\partial \boldsymbol{n}} - \int_{\partial B_R} \frac{\partial E}{\partial \boldsymbol{n}}$. It is easy to verify that $\int_{\partial B_R} \frac{\partial E}{\partial \boldsymbol{n}} = 1$, then $\langle h, 1\rangle = 0$. Consequently, from Lemma 2.10 [3], we deduce $w_0 = 0$ in Ω. Therefore, $w = 0$ and $z = 0$ in Ω.

b) *The case $q \geqslant n$*: There exists a constant c such that $w - \tilde{z} = c$ in $\mathbb{R}^n$ and $w = c$ on Γ. Then, $\Delta w_0 = 0$ in Ω and $w_0 = c + \langle h, 1\rangle E$ on Γ. Applying again (6), we obtain $\langle c + \langle h, 1\rangle E, \frac{\partial\lambda}{\partial \boldsymbol{n}}\rangle_\Gamma = 0$. Set that $\mu = c + \langle h, 1\rangle E$ on Γ. It is not difficult to see that $\mu \in W^{1-\frac{1}{r},r}(\Gamma)$ with any $r \in]\frac{n}{n-1}, n[$. Then there exists a unique $y \in W_0^{1,r}(\Omega)$ such that $\Delta y = 0$ in Ω and $y = \mu$ on Γ. Then, we deduce that $y - w_0 \in \mathcal{A}^{p,r}(\Omega)$. Thanks to the results for the case 2a) of this lemma, we have $y = w_0$, i.e., $w_0 \in W_0^{1,p}(\Omega) \cap W_0^{1,r}(\Omega)$. We can see that μ also belongs to $W^{1-\frac{1}{q},q}(\Gamma)$. Then there exists $\theta \in W_0^{1,q}(\Omega)$ such that $\Delta\theta = 0$ in Ω and $\theta = \mu$ on Γ. Then, $\theta - w_0 \in \mathcal{A}^{r,q}(\Omega)$. From the case 1), there exists a constant α such that $\theta - w_0 = \alpha(\lambda - 1)$ and we deduce that $w_0 \in W_0^{1,q}(\Omega)$. Consequently, the function $w \in W_0^{1,q}(\Omega)$ and since $w = c$ on Γ and from the characterization of $\mathcal{A}^q(\Omega)$, we can immediately deduce that $w = c\lambda$ and $z = c(\lambda - 1)$ in Ω.

3) *The case $p = \frac{n}{n-1}$*: Finally, let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \varphi = 1$ and $\mu = E * \varphi$. We know that $\mu \in L^{n,\infty}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ for any $r > n$ and $\nabla\mu \in L^{n/(n-1),\infty}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ for any $s > \frac{n}{n-1}$. The reasoning applies by replacing δ by φ and E by μ.

B. *If $n = 2$*: We know that there exists a unique $w_0 \in W_0^{1,p}(\mathbb{R}^2)$ satisfying $\Delta w_0 = h - \langle h, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} \Delta u_0$ in $\mathbb{R}^2$, where u_0 is defined by (3). Now we set $w = w_0 + \langle h, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} u_0$. Then $\Delta w = h$ in $\mathbb{R}^2$ and $w - \tilde{z}$ is harmonic. Proceeding as in the case A2 by distinguishing 2 cases $q \leqslant 2$ and $q > 2$, we can prove that $\mathcal{A}^{p,q}(\Omega) = \{0\}$ if $q \leqslant 2$ and $\mathcal{A}^{p,q}(\Omega) = \{c(\mu - u_0);\ c \in \mathbb{R}\}$ if $q > 2$ where $\mu \in \bigcap_{r>2} W_0^{1,r}(\Omega)$ is the unique solution of the problem (5). The proof is finished. $\square$

We complete this Note by a similar result for the three-dimensional Oseen equations with an analogous proof.

Theorem 2.2. *Let $1 < p < q < \infty$ and $\Omega \subset \mathbb{R}^3$ be an exterior domain with $C^{1,1}$ boundary.*

(i) *If $q < 4$, then $\mathcal{N}^{p,q}(\Omega) = \{(0, 0)\}$.*
(ii) *If $q \geqslant 4$, then $\mathcal{N}^{p,q}(\Omega) = \{(\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c);\ \boldsymbol{c} \in \mathbb{R}^3\}$ where $(\boldsymbol{\lambda}_c, \mu_c)$ is the unique solution of the following system*

$$-\Delta\boldsymbol{\lambda}_c + \frac{\partial\boldsymbol{\lambda}_c}{\partial x_1} + \nabla\mu_c = 0, \quad \operatorname{div}\boldsymbol{\lambda}_c = 0 \quad \text{in } \Omega, \quad \boldsymbol{\lambda}_c = \boldsymbol{c} \quad \text{on } \Gamma,$$

such that $\boldsymbol{\lambda}_c \in \bigcap_{r>4/3} \mathbf{X}_0^{1,r}(\Omega)$ and $\mu_c \in \bigcap_{r>3/2} L^r(\Omega)$. Moreover, we have $\boldsymbol{\lambda}_c \in \mathbf{L}^s(\Omega) \cap \mathbf{L}^\infty(\Omega)$ for all $s > 2$.

Here, the kernel $\mathcal{N}^{p,q}(\Omega)$ of the exterior Oseen system is defined by

$$\mathcal{N}^{p,q}(\Omega) = \left\{(\boldsymbol{u}, \pi) \in \left[\mathbf{X}_0^{1,p}(\Omega) + \mathbf{X}_0^{1,q}(\Omega)\right] \times \left[L^p(\Omega) + L^q(\Omega)\right], \mathbf{T}(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega,\ \boldsymbol{u} = 0 \text{ on } \Gamma\right\}$$

with $1 < p < q < \infty$. Besides, $\mathbf{X}_0^{1,p}(\Omega)$ and $\mathbf{T}(\boldsymbol{u}, \pi)$ are defined as follows

$$\mathbf{X}_0^{1,p}(\Omega) = \begin{cases} \boldsymbol{u} \in \mathbf{W}_0^{1,p}(\Omega) \cap \mathbf{L}^{4p/(4-p)}(\Omega) \text{ and } \frac{\partial \boldsymbol{u}}{\partial x_1} \in \mathbf{W}_0^{-1,p}(\Omega) & \text{if } 1 < p < 4, \\ \boldsymbol{u} \in \mathbf{W}_0^{1,p}(\Omega) \text{ and } \frac{\partial \boldsymbol{u}}{\partial x_1} \in \mathbf{W}_0^{-1,p}(\Omega) & \text{if } p \geqslant 4, \end{cases}$$

$$\mathbf{T}(\boldsymbol{u}, \pi) = \left(-\Delta\boldsymbol{u} + \frac{\partial\boldsymbol{u}}{\partial x_1} + \nabla\pi, -\operatorname{div}\boldsymbol{u}\right).$$

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 2003.
[2] C. Amrouche, V. Girault, J. Giroire, Weighted Sobolev spaces for the Laplace equation in $\mathbb{R}^n$, Journal de Mathématiques Pures et Appliquées 73 (6) (1994) 579–606.
[3] C. Amrouche, V. Girault, J. Giroire, Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces, Journal de Mathématiques Pures et Appliquées 76 (1) (1997) 55–81.