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A new duality transform ☆

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Abstract

Continuing our search for dualities in different classes of functions, which usually turn out to have an essentially unique form, depending on the class, we exhibit a natural class of functions for which there are exactly two different types of duality transforms. One is the well known Legendre transform, and the other is new. We study the new transform, give a simple geometric interpretation for it, and present some applications. ***To cite this article: S. Artstein-Avidan, V. Milman, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Une nouvelle transformée de dualité. Dans le cadre de notre étude de la dualité pour différentes classes de fonctions, souvent déterminée d'une façon unique par la classe, on exhibe une classe naturelle pour laquelle il y a exactement deux types de transformés de dualité. Une est la transformée de Legendre, et l'autre est nouvelle. Ces deux transformées ont des interprétations géométriques simples. On donne plusieurs applications des résultats. ***Pour citer cet article : S. Artstein-Avidan, V. Milman, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

On introduit le concept de « dualité abstraite » par :

Definition *(Dualité Abstraite).* On dira qu'une transformée $\mathcal{T}$ génère une transformée de dualité sur un ensemble $\mathcal{S}$ de fonctions sur $\mathbb{R}^n$ si on a les proprietés suivantes :

1. Pour toute $f \in \mathcal{S}$ on a $\mathcal{T}\mathcal{T} f = f$.
2. Pour tout couple de fonctions $f, g \in \mathcal{S}$ telles que $f \leqslant g$, on a $\mathcal{T} f \geqslant \mathcal{T} g$.

Cette definition est motivée par des résultats récents. Dans [3] on a demontré que sur une classe de fonctions convexes semi-continues sur $\mathbb{R}^n$, il y a, essentiellement, une seule dualité – la transformée de Legendre qui est bien

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connue. Plus précisement, on désigne la classe des fonctions convexes s.c.i. $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ par $\mathrm{Cvx}(\mathbb{R}^n)$. On note $\langle\cdot,\cdot\rangle$ le produit scalaire usuel sur $\mathbb{R}^n$. On rappelle la definition classique de la transformée de Legendre pour des fonctions $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ par :

$$(\mathcal{L}\phi)(x) = \sup_y \bigl(\langle x, y\rangle - \phi(y)\bigr). \tag{1}$$

On a demontré dans [3] :

Theorème A *Supposons qu'une transformée* $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ (*définie sur tout* $\mathrm{Cvx}(\mathbb{R}^n)$) *satisfait les conditions de Dualité Abstraite. Alors,* $\mathcal{T}$ *est essentiellement la transformée de Legendre classique, c.a.d. qu'il existe une constante* $C_0 \in \mathbb{R}$, *un vecteur* $v_0 \in \mathbb{R}^n$ *et une transformation symétrique* $B \in GL_n$ *tels que*

$$(\mathcal{T}\phi)(x) = (\mathcal{L}\phi)(Bx + v_0) + \langle x, v_0\rangle + C_0.$$

Des résultats du analogues ont été demontrés très récemment. Avant d'avoir achevé la démonstration du Théorème A, Böröczky et Schneider [5] ont demontré que sur une classe de corps convexes dans $\mathbb{R}^n$ il y a (essentiellement) une seule dualité (classique). Plus précisément, notons $\mathcal{K}_{(0)}(\mathbb{R}^n)$ la classe des corps convexes et compacts dans $\mathbb{R}^n$ dont l'intérieur contient 0. On a :

Theorème B *(Böröczky–Schneider). Soit* $n \geqslant 2$. *Supposons qu'on ait une transformée* $\mathcal{T} : \mathcal{K}_{(0)}(\mathbb{R}^n) \to \mathcal{K}_{(0)}(\mathbb{R}^n)$ (*définie sur tout* $\mathcal{K}_{(0)}(\mathbb{R}^n)$) *satisfaisant* :

1. $\mathcal{T}\mathcal{T}K = K$.
2. $K_1 \subset K_2$ *implique* $\mathcal{T}K_1 \supset \mathcal{T}K_2$.

Alors, $\mathcal{T}$ *est essentiellement la transformée de polarité* $(\cdot)^\circ$ *habituelle qui envoie le corps* K *sur le corps polaire* K° *défini par* $K^\circ = \{x\colon \sup_{y\in K}\langle x, y\rangle \leqslant 1\}$. *Plus précisément, il existe une transformation symétrique* $B \in GL_n$, *telle que pour tout* K, $\mathcal{T}K = BK^\circ$.

Dans [4] nous avons demontré la même résultat pour la classe plus grande des corps convexes fermés qui contiennent 0 (peut-être sur la frontière). On remarque qu'aucun de ces deux theorèmes n'implique l'autre. Si on note $\mathcal{K}_0(\mathbb{R}^n)$ la classe des ensembles convexes fermés dans $\mathbb{R}^n$ qui contiennent 0, on a

Theorème C *(Voir [4].) Soit* $n \geqslant 2$. *Supposons qu'on ait une transformée* $\mathcal{T} : \mathcal{K}_0(\mathbb{R}^n) \to \mathcal{K}_0(\mathbb{R}^n)$ *satisfaisant* (1.) *et* (2.) *du Theorème* 2 *ci-dessus, alors il existe* $B \in GL_n$, *symétrique telle que pour tout* K, $\mathcal{T}K = BK^\circ$.

Le même énoncé est vrai dans la classe des espaces normés sur $\mathbb{R}^n$ (voir [6] et [5]), pour des cônes convexes [8] et de nombreux autres cas (voir [2]). On remarque que dans tous les théorèmes ci-dessus on peut remplacer la condition d'involution par la condition plus faible que la transformée est inversible et que l'inverse renverse également l'ordre, et on obtient le même resultat modulo des termes linéaires.

Il est assez evident de dire que sur une classe naturelle de fonctions il existe (essentiellement) une unique transformée de dualité. Il est donc surprenant de constater qu'un changement très "naturel" de la classe de fonctions introduit une (nouvelle !) dualité supplémentaire.

Soit $\mathrm{Cvx}_0(\mathbb{R}^n)$ la classe des fonctions convexes s.c.i. $f : \mathbb{R}^n \to [0, \infty]$, qui prennent la valeur 0 en 0. La transformée de Legendre opère de façon invariante sur cette classe et ainsi représente une dualité sur la classe (conformément à la définition de dualité abstraite donnée ci-dessus). On considère la transformée suivante :

$$(\mathcal{A}f)(x) = \begin{cases} \sup_{\{y\in\mathbb{R}^n\colon f(y)>0\}} \frac{\langle x,y\rangle - 1}{f(y)} & \text{si } x \in \{f(y) = 0\}^\circ, \\ +\infty & \text{si } x \notin \{f(y) = 0\}^\circ \end{cases}$$

(avec la convention $\sup\emptyset = 0$).

Il est évident que $\mathcal{A}$ préserve l'ordre, et on peut démontrer que $\mathcal{A}$ est une involution (cela se déduit par exemple de l'interprétation géométrique de la transformée ci-dessous). C'est donc encore une "dualité abstraite" sur cette classe. On observe d'autres propriétés de cette dualité. Pour toute norme $\|\cdot\|$, on a $(\mathcal{A}\|\cdot\|)(y) = \|y\|^*$, où $\|x\|^* = \sup\{\langle x, y\rangle\colon \|y\| \leqslant 1\}$ est la norme duale. En fait, pout tout puissance $p \geqslant 1$ on a $\mathcal{A}(\|\cdot\|^p) = \frac{1}{p\cdot q^{p-1}}(\|x\|^*)^p$.

On peut montrer que cette deuxième dualité dans la classe $\mathrm{Cvx}_0(\mathbb{R}^n)$ est la seule autre possibilité, plus précisement on a :

Theorème 1. *Soit $n \geqslant 2$. Toute involution sur $\mathrm{Cvx}_0(\mathbb{R}^n)$ qui renverse l'ordre est soit de la forme $f \mapsto (\mathcal{L}f) \circ B$ soit ou de la forme $f \mapsto C_0(\mathcal{A}f) \circ B$ pour un $B \in GL_n$ symétrique et $C_0 > 0$.*

1. Introduction

The concept of "abstract duality" was introduced by the authors as follows (see [2]):

Definition *(Abstract Duality).* We will say that a transform $\mathcal{T}$ generates a duality transform on a set of functions $\mathcal{S}$ on $\mathbb{R}^n$ if the following two properties are satisfied:

1. For any $f \in \mathcal{S}$ we have $\mathcal{T}\mathcal{T}f = f$.
2. For any two functions in $\mathcal{S}$ satisfying $f \leqslant g$ we have that $\mathcal{T}f \geqslant \mathcal{T}g$.

The definition was motivated by a few recent results, which demonstrated that in several classes of functions, there is essentially only one duality transform (we quote some of these results below). In this note we present a natural class of functions (non-negative, convex lower-semi-continuous functions on $\mathbb{R}^n$ with $f(0) = 0$) for which there are exactly *two* essentially different duality transforms. One of these transforms is the well known Legendre transform (2), and the other is new, and we call it $\mathcal{A}$. Its definition is given in (3) below, and a simple geometric interpretation of it is given in Section 4. Some interesting consequences follow, such as the existence of an involutive order preserving transformation which is not the identity, and the extension of some geometric operations for functions.

Let us begin by citing some earlier results. In [3] we proved that on the class of lower-semi-continuous convex functions on $\mathbb{R}^n$ there is, essentially, only one duality: the well-known Legendre transform. More precisely, denote the class of lower-semi-continuous convex functions $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ by $\mathrm{Cvx}(\mathbb{R}^n)$. Denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^n$. Recall the definition of the classical Legendre transform $\mathcal{L}$ defined for functions $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ by

$$(\mathcal{L}\phi)(x) = \sup_y \big(\langle x, y \rangle - \phi(y)\big). \tag{2}$$

We proved in [3] that a transform $\mathcal{T} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ (defined on the whole domain $\mathrm{Cvx}(\mathbb{R}^n)$) which satisfies "abstract duality" must be essentially the classical Legendre transform (Theorem A above). Namely, there exist a constant $C_0 \in \mathbb{R}$, a vector $v_0 \in \mathbb{R}^n$, and a symmetric transformation $B \in GL_n$ such that

$$(\mathcal{T}\phi)(x) = (\mathcal{L}\phi)(Bx + v_0) + \langle x, v_0 \rangle + C_0.$$

More results in this spirit were proved during the last year. Still before we finished our proof of Theorem 1, and answering our question, Böröczky and Schneider [5] proved that for $n \geqslant 2$, on the class of compact convex bodies in $\mathbb{R}^n$, with 0 in the interior, (denoted $\mathcal{K}_{(0)}(\mathbb{R}^n)$) there is (again essentially) only one duality, the classical one, defined by $K^\circ = \{x\colon \sup_{y \in K} \langle x, y \rangle \leqslant 1\}$ (see Theorem B above). In [4] we proved the same for the larger class of all closed convex sets containing 0, possibly at the boundary (denoted $\mathcal{K}_0(\mathbb{R}^n)$), which is given as Theorem C above. The same is also true for the class of normed spaces in $\mathbb{R}^n$, i.e. for symmetric convex bodies (combining results of Gruber [6] and Böröczky–Schneider [5]), for convex cones [8] and in many other examples (see [2]). We remark that in all above theorems the condition of involution can be replaced by the weaker condition that the transform is invertible and its inverse is also order reversing, with the same result except for several extra linear changes.

It already seemed almost routine that on natural classes of functions there exists (essentially) one unique duality transform. However, to our surprise, it turned out that a very "natural" change of the class of functions brought with it an additional (new!) duality.

2. Another duality

Let $\mathrm{Cvx}_0(\mathbb{R}^n)$ denote the class of convex l.s.c. $f:\mathbb{R}^n \to [0,\infty]$, which take the value 0 at 0. Clearly, the Legendre transform acts invariantly on this class, and so represents a duality on it (with respect to our definition of "abstract duality" given above). Consider the following transform:

$$(\mathcal{A}f)(x) = \begin{cases} \sup_{\{y\in\mathbb{R}^n:\ f(y)>0\}} \frac{\langle x,y\rangle - 1}{f(y)} & \text{if } x \in \{f(y)=0\}^\circ, \\ +\infty & \text{if } x \notin \{f(y)=0\}^\circ \end{cases} \tag{3}$$

(with the convention $\sup \emptyset = 0$). Note that although $\mathcal{A}$ implicitly depends on the dimension n, we do not complicate notation and use the same letter $\mathcal{A}$ for all dimensions.

Obviously, $\mathcal{A}$ is order reversing, and one may show (it follows e.g. from the geometric interpretation below) that it is an involution. So, it is again an "abstract duality" on $\mathrm{Cvx}_0(\mathbb{R}^n)$. Let us observe some other properties of $\mathcal{A}$. For any norm $\|\cdot\|$, we have that $(\mathcal{A}\|\cdot\|)(y) = \|y\|^*$, where $\|x\|^* = \sup\{\langle x,y\rangle:\ \|y\| \leqslant 1\}$ is the dual norm. The same is true also when $\|\cdot\|$ is a generalized norm, that is, it need not be symmetric and may assume the values 0 and $+\infty$. From here onwards we let $\|\cdot\|$ stand for such a "generalized norm", that is, a positively-homogeneous convex function with values in $\mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ (so that its unit ball is some closed convex set K, possibly unbounded, containing 0, possibly at its boundary, and $\|\cdot\| = \|\cdot\|_K$ is called the Minkowski functional of the body K).

In fact, for every power $p \geqslant 1$ we have that $\mathcal{A}(\|\cdot\|^p) = \frac{1}{p\cdot q^{p-1}}(\|x\|^*)^p$, where $\frac{1}{p} + \frac{1}{q} = 1$ (and where $\infty^0 = 1$). In particular, since for $a > 0$, $\mathcal{A}(af)(y) = (1/a)(\mathcal{A}f)(y)$, we have that and $\mathcal{A}(\|\cdot\|^2/2)(y) = (\|y\|^*)^2/2$ for any generalized norm. That is, $\mathcal{A}$ coincides with $\mathcal{L}$ on the subclass of 2-homogeneous functions, but for p-homogeneous functions $p \neq 2$ there is a dramatic difference between the two duality transforms.

We also remark that for a convex function $f(t)$ on $\mathbb{R}^+$, $f(0) = 0$ and for any norm $\|\cdot\|$ we have that $\mathcal{A}(f(\|\cdot\|)) = (\mathcal{A}f)(\|\cdot\|^*)$ where the first transform is on $\mathbb{R}^n$ and the second is the one-dimensional one, on the half-line $\mathbb{R}^+$ (one may define $\mathcal{A}$ on a cone instead of on the full linear space). Note that the same is true for $\mathcal{L}$, namely $\mathcal{L}(f(\|\cdot\|)) = (\mathcal{L}f)(\|\cdot\|^*)$.

It turns out that this second form of duality for the class $\mathrm{Cvx}_0(\mathbb{R}^n)$ is the only other option:

Theorem 1. *Let $n \geqslant 2$. Any order reversing involution of* $\mathrm{Cvx}_0(\mathbb{R}^n)$ *is either of the form $f \mapsto (\mathcal{L}f)\circ B$ or $f \mapsto C_0(\mathcal{A}f)\circ B$ for some symmetric $B \in GL_n$ and $C_0 > 0$.*

The condition of involution may be replaced by the weaker condition that both $\mathcal{A}$, which we assume to be 1-1 and onto, and its inverse, are order reversing, in which case we get the same conclusion but where B need not be symmetric.

A few more remarks are in order:

First, the one dimensional version is true as well, however by trivial reasons there are eight possible dualities, with two "free" positive constants.

Secondly, the complete description of order reversing maps on Cvx_0 provides also the complete description of order preserving maps, and, again, there are (in dimension > 1) exactly two essentially different order preserving maps. One is the 'identity', namely of the form $f \mapsto C_0 f \circ B$ and the second is $f \mapsto C_0\mathcal{L}\mathcal{A}f \circ B$ for some $B \in GL_n$. To prove this we take any order preserving transform and compose it with the Legendre transform. We get an invertible transformation which is order reversing, and so is its inverse. Thus, by the remark after Theorem 1 it must be essentially either the Legendre transform (in which case the original transform must have been essentially the identity) or the new $\mathcal{A}$ transform, in which case the original transform is essentially of the form $f \to \mathcal{L}\mathcal{A}f$. The formula for this order preserving transform can be then computed, and has the following form (we give it for a convex l.s.c. $f:\mathbb{R}^n \to [0,\infty)$, with $f(0) = 0$ which is not identically 0; the constant function 0 is mapped to 0, and for a function with values which are $+\infty$, simply approximate it by an increasing sequence). The function $g = \mathcal{L}\mathcal{A}f$ is given as follows:

$$g(y) = \inf\bigl\{1/f(x):\ y = tx/f(x),\ 0 \leqslant t \leqslant 1\bigr\}.$$

(Here the infimum of an empty set is $+\infty$, and $0/f(0)$ is understood in limit sense.)

Note that this transform, although very different from the identity, still has the very special property, that it acts 'ray-wise', i.e., its values on a given ray depend only on the values of f on the ray. It might not be a-priori clear why the resulting function g is convex. This will, however, be immediate from the geometric interpretation below.

Our third remark (which actually follows from the second) is that our two dualities commute: $\mathcal{L}\mathcal{A} = \mathcal{A}\mathcal{L}$.

3. The second duality on a subclass of Log-Concave functions

The Legendre transform $\mathcal{L}$ led to a duality transform on $\mathrm{LC}(\mathbb{R}^n)$ (the class of upper semi-continuous log-concave functions) in a very natural way (see [1]): for $f \in \mathrm{LC}(\mathbb{R}^n)$ we defined $f^\circ = \mathrm{e}^{-\mathcal{L}(-\log f)}$ to be *the* dual function. (By Theorem A this is essentially the only transform satisfying abstract duality for this class.)

The transform $\mathcal{A}$ leads to a new (second) duality on an important subclass. Let us define the subclass

$$\mathrm{LC}_1(\mathbb{R}^n) = \{f \in \mathrm{LC}(\mathbb{R}^n):\ f(0) = 1 \text{ and } 0 \leqslant f(x) \leqslant 1\}.$$

Note that for $f \in \mathrm{LC}_1(\mathbb{R}^n)$ also $f^\circ \in \mathrm{LC}_1(\mathbb{R}^n)$, and satisfies, of course, abstract duality. The second duality on this class is given by:

$$f^\odot = \mathrm{e}^{-\mathcal{A}(-\log f)}, \quad \text{i.e.} \quad -\log f^\odot = \mathcal{A}(-\log f).$$

Then we get, naturally, that $(f^\odot)^\odot = f$ and that $(\cdot)^\odot$ is order reversing. Also, from the third remark in the previous section, $\mathcal{A}(-\log f^\circ) = \mathcal{L}(-\log f^\odot)$ so that $(f^\odot)^\circ = (f^\circ)^\odot$.

For a closed convex set K, with $0 \in K$ we have the property $(1_K)^\odot = 1_{K^\circ}$, which means that this is indeed an extension of standard duality, a desired property which, it follows from Theorem 1, cannot hold on the whole class $\mathrm{LC}(\mathbb{R}^n)$. For *any* (generalized) norm $\|\cdot\|$ and $p \geqslant 1$,

$$\left(\exp\left(-a_p\|x\|^p\right)\right)^\odot = \exp\left(-a_p\left(\|x\|^*\right)^p\right)$$

where $a_p = (\frac{1}{p\cdot q^{p-1}})^{1/2}$, $\frac{1}{q} + \frac{1}{p} = 1$.

4. Geometric interpretations

There are two different ways to associate to a function in $\mathrm{Cvx}_0(\mathbb{R}^n)$ a closed convex set in $\mathbb{R}^{n+1}$ which is contained in the upper half-space $\mathbb{R}^n \times \mathbb{R}_{\geqslant 0}$, and contains the ray $0 \times \mathbb{R}_{\geqslant 0}$. One natural way is to consider the epi-graph of the function $\mathrm{epi}(f) = \{(x, r):\ r \geqslant f(x)\}$. Another is to extend f from the hyperplane $\{(x, 1):\ x \in \mathbb{R}^n\}$ homogeneously for (x, r) with $r > 0$, and to take the closure of the unit ball of the generalized norm obtained by this procedure. This gives:

$$K_f = \overline{\{(x, y) \in \mathbb{R}^n \times \mathbb{R}:\ 0 < y \text{ and } f(x/y) \leqslant 1/y\}}.$$

Note that duality-and-reflection acts invariantly on this class of convex subsets of $\mathbb{R}^{n+1}$, namely if K is in the class then so is $D(K) = \{(x, -r):\ (x, r) \in K^\circ\}$.

It is an easy exercise to verify that the reflection of the polar of $\mathrm{epi}(f)$ with respect to $\mathbb{R}^n$ is the epi-graph of $\mathcal{A}f$, that is

$$\mathrm{epi}(\mathcal{A}f) = \{(x, -r) \in \mathbb{R}^n \times \mathbb{R}_{\geqslant 0}:\ (x, r) \in (\mathrm{epi}(f))^\circ\}.$$

It is also not difficult to check that the same is true for the second description, that is,

$$K_{(\mathcal{A}f)} = \{(x, -r) \in \mathbb{R}^n \times \mathbb{R}_{\geqslant 0}:\ (x, r) \in (K_f)^\circ\}.$$

In particular, the involution property of $\mathcal{A}$ follows immediately.

This understanding also gives a description for the order-preserving transformation, namely the relation $\mathrm{epi}(\varphi) = K_f$ gives an order preserving correspondence $f \leftrightarrow \varphi$ (which is an involution), and this is precisely $\varphi = \mathcal{L}\mathcal{A}f$. That is,

$$\mathcal{L}\mathcal{A}f = \mathrm{epi}^{-1}(K_f) \quad \text{and} \quad K_{(\mathcal{L}\mathcal{A}f)} = \mathrm{epi}(f).$$

5. The Support map and the Minkowski map

In this section we show how two central operations of classical convexity, namely the support functionals and the Minkowski functionals, which are very geometric constructions, may be uniquely defined in the language of "order preserving"/"order reversing" maps. Through this understanding they can be extended to the functional class $\mathrm{LC}(\mathbb{R}^n)$

(or its subclass $\mathrm{LC}_1(\mathbb{R}^n)$) in a natural way. It has been realized recently that the extension of geometric notions and results is an important goal in Asymptotic Geometric Analysis and is now called “Geometrization of Probability” (see [7]). We start with analyzing the support-operator S.

It is classical and well known that for a body $K \in \mathcal{K}_0(\mathbb{R}^n)$ there corresponds a convex positively-homogeneous function on $\mathbb{R}^n$ which is its support function $h_K := \|\cdot\|_K^*$. We denote $S(K) = h_K$ so that $S : \mathcal{K}(\mathbb{R}^n) \to \mathcal{H}^n$ is an order preserving transformation (with respect to the partial order of inclusion for sets and the partial order of point-wise inequality on functions). Here $\mathcal{K}(\mathbb{R}^n)$ can stand for either $\mathcal{K}_0(\mathbb{R}^n)$ or $\mathcal{K}_{(0)}(\mathbb{R}^n)$, and $\mathcal{H}^n$ will stand for either convex positively-homogeneous functions on $\mathbb{R}^n$ with values in $\mathbb{R}^+$ or such functions with values in $\mathbb{R}^+ \cup \{+\infty\}$.

The Minkowski operator M is similarly defined $M : \mathcal{K}(\mathbb{R}^n) \to \mathcal{H}^n$. It maps the body K to the positively-homogeneous function $\|\cdot\|_K$. (Thus we have of course $M(K) = S(K^\circ)$ and vice-versa). The Minkowski map is, thus, an order reversing transformation. Both S and M are essentially unique (in the following theorem notation is slightly abused).

Theorem 2. *Any mapping $\mathcal{T} : \mathcal{K}(\mathbb{R}^n) \to \mathcal{H}^n$ which preserves the (partial) order must be, up to a linear change, the ‘support’ map S defined above.*

Any mapping $\mathcal{T} : \mathcal{K}(\mathbb{R}^n) \to \mathcal{H}^n$ which reverses the (partial) order must be, up to a linear change, the Minkowski map M defined above.

We note that in fact two theorems are stated here, one for $\mathcal{K}_{(0)}(\mathbb{R}^n)$ for which the above result follows directly by Theorem B and [5], and another for $\mathcal{K}_0(\mathbb{R}^n)$ for which the above holds by Theorem C and other results of [4].

Next, one may extend the operation of support function from the class $\mathcal{K}_0(\mathbb{R}^n)$ to the class $\mathrm{LC}(\mathbb{R}^n)$ (where the embedding $\mathcal{K}(\mathbb{R}^n) \subset \mathrm{LC}(\mathbb{R}^n)$ is simply $K \to 1_K$). The extension is given by $S(f) = \mathcal{L}(-\log f)$. Again this is an essentially unique order preserving mapping, this time between $\mathrm{LC}(\mathbb{R}^n)$ and $\mathrm{Cvx}(\mathbb{R}^n)$ as follows from Theorem A and its relatives. More interestingly, our theorems imply that M does not admit an order reversing extension to $\mathrm{LC}(\mathbb{R}^n)$, and has a *unique* extension to a transform from $\mathrm{LC}_1(\mathbb{R}^n)$ to $\mathrm{Cvx}_0(\mathbb{R}^n)$ which reverses the order of functions.

Indeed, by Theorem 1 there are essentially only two transforms between $\mathrm{LC}_1(\mathbb{R}^n)$ and $\mathrm{Cvx}_0(\mathbb{R}^n)$ which preserve order, one of them is $f \to \mathcal{L}(-\log f)$, and the other is $f \to \mathcal{A}(-\log f)$. Thus, an order reversing transformation between $\mathrm{LC}_1(\mathbb{R}^n)$ and $\mathrm{Cvx}_0(\mathbb{R}^n)$ is either essentially of the form $f \to \mathcal{L}\mathcal{L}(-\log f) = -\log f$, which does not extend the operation M from the class of convex sets, or essentially of the form $f \to \mathcal{L}\mathcal{A}(-\log f)$, which restricted to indicator functions of bodies in $\mathcal{K}_0(\mathbb{R}^n)$, is equal to M (and also equals $\mathcal{A}S(f)$). Similarly by Theorem A the only order reversing transform between $\mathrm{Cvx}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$ is $f \to -\log f$, which does *not* extend M.

We end with a few more examples of the behavior of these transforms:

(i) $S(1_K) = \mathcal{L}(-\log 1_K) = \|\cdot\|_K^*$, and $M(1_K) = \|\cdot\|_K$,
(ii) $S(\mathrm{e}^{-\|\cdot\|}) = -\log(1_{K^\circ})$ and $M(\mathrm{e}^{-\|\cdot\|}) = -\log(1_K)$.
(iii) In general, $M(f) = \mathcal{A}S(f) = \mathcal{A}\mathcal{L}(-\log f) = \mathcal{L}\mathcal{A}(-\log f) = Sf^{\odot}$.

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