Available online at www.sciencedirect.com
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C. R. Acad. Sci. Paris, Ser. I 346 (2008) 1175–1180
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Dynamical Systems/Ordinary Differential Equations

Invariant manifold theory via generating maps

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Received 21 May 2008; accepted after revision 29 September 2008

Available online 16 October 2008

Presented by Étienne Ghys

Abstract

We present a synthetic approach to invariant manifold theorems, based upon the notion of a generating map. ***To cite this article: M. Chaperon, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Applications génératrices et variétés invariantes. Nous présentons une approche synthétique de la théorie des variétés invariantes, fondée sur la notion d'application génératrice. ***Pour citer cet article : M. Chaperon, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Une *correspondance* d'un ensemble Z dans lui-même est une application h de Z dans l'ensemble $\mathcal{P}(Z)$ des parties de Z. Elle est déterminée par son *graphe* $\operatorname{graph}(h) := \{(z,z') \in Z^2\colon\ z' \in h(z)\}$, qui peut être *n'importe quelle* partie de Z^2 (formellement, une correspondance est donc une relation binaire). Bien sûr, une application $f : Z \to Z$ s'identifie à la correspondance $z \mapsto \{f(z)\}$.

Une *orbite de longueur* $n \in \mathbf{N}$ de h est une suite finie $(z_0,\dots,z_n) \in Z^{n+1}$ vérifiant (1). L'*itérée* n-ième h^n de h est la correspondance de Z dans lui-même dont le graphe est l'ensemble des $(z,z') \in Z^2$ tels qu'il existe une orbite $(z_0,\dots,z_n)$ de longueur n de h avec $z_0 = z$ et $z_n = z'$. En particulier, h^0 est l'identité et $h^1 = h$. Une *orbite* de h est une suite $(z_k)_{k\in\mathbf{N}}$ dans Z vérifiant (1) pour tout $n \in \mathbf{N}$.

L'*inverse* de h est la correspondance h^{-1} de Z dans lui-même dont le graphe est l'image de $\operatorname{graph}(h)$ par l'involution $(z,z') \mapsto (z',z)$ de Z^2. Autrement dit, $h^{-1}(z') := \{z\colon\ z' \in h(z)\}$ (la correspondance inverse d'une application non bijective de Z dans lui-même n'est donc pas une application). Pour tout $n \in \mathbf{N}$, on pose $h^{-n} := (h^{-1})^n$.

Lorsque Z est un produit $X \times Y$, nous dirons que la correspondance h admet l'*application génératrice* $H = (F,G) : Z \to Z$ quand le graphe de h est l'ensemble des $(x,y,x',y') \in Z^2$ vérifiant (2). Cela revient à dire que, pour tout $(x,y') \in Z$, il existe une unique orbite (z_0,z_1) de longueur 1 de h telle que la première composante de z_0 soit x

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.09.030

et que la seconde composante de z_1 soit y'. On explique ci-après comment les quatre théorèmes suivants se prouvent, en renvoyant à [2,1] pour les exemples et les applications :

Théorème 1. *Étant donnés deux espaces métriques non vides X, Y, on munit Z de sa distance d'espace produit $d((x,y),(x',y')) := \max\{d(x,x'), d(y,y')\}$. Soit h une correspondance de Z dans lui-même, admettant une application génératrice lipschitzienne $H = (F,G)$ vérifiant* (3). *Si Y est complet, alors, pour tout entier $n > 0$, la correspondance h^n a une application génératrice $H_n = (F_n, G_n)$ et, quels que soient $z = (x,y)$ et $z' = (x',y')$ dans Z, les inégalités* (4)–(5) *sont vérifiées. Pour chaque $z = (x,y) \in Z$, il existe une seule orbite $(z_0, \ldots, z_n)$ de longueur n de h telle que x soit la première composante de z_0 et y la seconde composante de z_n ; en particulier, la seconde composante de z_{n-1} s'écrit $y_{n-1} = A_{n-1}(z)$, ce qui définit une application $A_{n-1} : Z \to Y$ vérifiant* (6).

Théorème 2. *Sous les hypothèses du Théorème* 1*, on suppose Y complet et $\mu < 1$. Alors, pour tout $x \in X$ et $1 \leqslant \kappa < \mu^{-1}$, il existe une unique orbite $(x_n, y_n)_{n\in\mathbf{N}}$ de h telle que $x_0 = x$ et que $(y_n)_{n\in\mathbf{N}}$ appartienne à l'espace $\mathcal{Y}_\kappa$ des suites $(y_n)_{n\in\mathbf{N}}$ dans Y vérifiant* $\sup_n \kappa^{-n} d(y_n, y) < \infty$ *pour un* (*et donc tout*) *$y \in Y$. Si Y est borné, comme $\mathcal{Y}_\kappa = Y^{\mathbf{N}}$, c'est l'unique orbite $(x_n, y_n)_{n\in\mathbf{N}}$ de h telle que $x_0 = x$. En désignant y_0 par $\varphi(x)$, l'application $\varphi : X \to Y$ vérifie $\varphi(x) = \lim_{n\to\infty} G_n(x,y)$ pour chaque $y \in Y$ et tout $x \in X$, d'où $\operatorname{Lip}\varphi \leqslant \mu$ d'après* (5)*. Son graphe W_s est invariant par h en ce sens que $h^{-1}(W_s) = W_s$, et $W_s \ni z \mapsto h(z) \cap W_s$ est une* application *lipschitzienne $h_s : W_s \to W_s$, telle que $\operatorname{Lip} h_s \leqslant \lambda$. Quand Y est borné, W_s est l'ensemble* (12) *des $z \in Z$ tels qu'il existe une orbite (z_n) de h avec $z_0 = z$.*

Théorème 3. *Étant données deux variétés de Finsler à coins X, Y de classe C^r, $r \geqslant 1$, soit h une correspondance de $Z := X \times Y$ dans lui-même, admettant une application génératrice $H = (F,G)$ de classe C^r telle qu'il existe des constantes positives λ, μ vérifiant* (14) *et des fonctions $\alpha, \beta : Z \to \mathbf{R}_+$ satisfaisant à* (15), (16) *et* (17) *pour tout $z = (x,y) \in Z$ et tout $\delta z = (\delta x, \delta y) \in T_z Z$. Si Y est complète, alors, pour tout entier $n > 0$, la correspondance h^n a une application génératrice $H_n = (F_n, G_n)$ de classe C^r. Pour tout $z = (x,y) \in Z$, il existe une* unique *orbite $(z_0, \ldots, z_n)$ de longueur n de h telle que, en posant $z_j = (x_j, y_j)$, on ait $x_0 = x$ et $y_n = y$; en particulier, $y_{n-1} = A_{n-1}(z)$ et l'application $A_{n-1} : Z \to Y$ ainsi définie est C^r. Pour tout $\delta z = (\delta x, \delta y) \in T_z Z$, en posant $v_j := (x_j, y_{j+1})$ pour $0 \leqslant j \leqslant n-1$, les inégalités* (18), (19) *et*[1] (20) *sont vérifiées.*

Théorème 4. *Sous les hypothèses du Théorème* 3*, on suppose Y complète et $\beta_1 := \sup \beta(Z) < 1$. Alors, pour tout $x \in X$ et $1 \leqslant \kappa < \beta_1^{-1}$, il existe une unique orbite $(x_n, y_n)_{n\in\mathbf{N}}$ de h telle que $x_0 = x$ et que $(y_n)_{n\in\mathbf{N}}$ appartienne à l'espace $\mathcal{Y}_\kappa$ du Théorème* 1*. Si Y est borné, c'est donc l'unique orbite $(x_n, y_n)_{n\in\mathbf{N}}$ de h telle que $x_0 = x$. En désignant y_0 par $\varphi(x)$, on définit une application $\varphi : X \to Y$ telle que $\varphi(x) = \lim_{n\to\infty} G_n(x,y)$ pour chaque $y \in Y$ et tout $x \in X$, donc $\operatorname{Lip}\varphi \leqslant \mu$ par* (19)*. Le graphe W_s de φ est invariant par h en ce sens que $h^{-1}(W_s) = W_s$, et $W_s \ni z \mapsto h(z) \cap W_s$ est une* application *localement lipschitzienne $h_s : W_s \to W_s$, globalement Lipschitzienne pour* $\sup \alpha(Z) < \infty$*. Quand Y est borné, W_s est l'ensemble* (12) *des $z \in Z$ tels qu'il existe une orbite (z_n) de h vérifiant $z_0 = z$.*

Les Théorèmes 1 et 2 reprennent pour l'essentiel la situation considérée dans l'article [2], dont les autres résultats sont justiciables du même traitement. Les Théorèmes 3 et 4 contiennent la théorie de l'hyperbolicité normale [4,6], la différentiabilité de φ s'établissant par exemple comme dans [1] pour $\sup_{z\in Z} \alpha(z)\beta(z) < 1$.

1. Introduction and definitions

A *correspondence* of a set Z into itself is a map h of Z into the set $\mathcal{P}(Z)$ of subsets of Z. It is determined by its *graph* $\operatorname{graph}(h) := \{(z,z') \in Z^2 \colon z' \in h(z)\}$, which can be *any* subset of Z^2 (thus, a correspondence is just a binary relation). Of course, a map $f : Z \to Z$ is identified to the correspondence $z \mapsto \{f(z)\}$.

An *orbit of length $n \in \mathbf{N}$* of h is a finite sequence $(z_0, \ldots, z_n) \in Z^{n+1}$ satisfying

$$z_{k+1} \in h(z_k) \quad \text{for } 0 \leqslant k < n. \tag{1}$$

[1] En convenant que $\alpha(v_{n-2}) \cdots \alpha(v_0) = 1$ si $n = 1$.

The n-th *iterate* h^n of h is the correspondence of Z into itself whose graph is the set of those $(z, z') \in Z^2$ such that there exists an orbit $(z_0, \dots, z_n)$ of length n of h with $z_0 = z$ and $z_n = z'$. In particular, h^0 is the identity map and $h^1 = h$. An *orbit* of h is a sequence $(z_k)_{k\in\mathbf{N}}$ in Z satisfying (1) for all $n \in \mathbf{N}$.

The *inverse* of h is the correspondence h^{-1} of Z into itself whose graph is the image of graph(h) by the involution $(z, z') \mapsto (z', z)$ of Z^2. In other words, $h^{-1}(z') := \{z\colon z' \in h(z)\}$ (the inverse correspondence of a nonbijective map of Z into itself is not a map). For every $n \in \mathbf{N}$, we let $h^{-n} := (h^{-1})^n$.

If Z is a product $X \times Y$, the correspondence h admits the *generating map* $H = (F, G) : Z \to Z$ when the graph of h is the set of those $(x, y, x', y') \in Z^2$ which satisfy

$$x' = F(x, y') \quad \text{and} \quad y = G(x, y'). \tag{2}$$

This means exactly that, for each $(x, y') \in Z$, there exists a unique orbit (z_0, z_1) of length one of h such that the first component of z_0 is x and the second component of z_1 is y'.

Hypothesis. Throughout the sequel, X, Y are nonempty metric spaces and h is a correspondence of $Z := X \times Y$ into itself, admitting a generating map $H = (F, G)$.

2. The "absolute" case in the Lipschitz category[2]

Hypothesis. (See [2].) Endowing Z with the distance $d((x, y), (x', y')) := \max\{d(x, x'), d(y, y')\}$, we assume that F, G are Lipschitzian and

$$\lambda\mu < 1, \quad \lambda := \operatorname{Lip} F, \quad \mu := \operatorname{Lip} G. \tag{3}$$

Theorem 1. *If Y is complete, then, for every positive integer n, the correspondence h^n has a generating map $H_n = (F_n, G_n)$ such that, for all $z = (x, y)$ and $z' = (x', y')$ in Z,*

$$d\big(F_n(z), F_n(z')\big) \leqslant \max\big\{\lambda^n d(x, x'), \lambda d(y, y')\big\}, \tag{4}$$

$$d\big(G_n(z), G_n(z')\big) \leqslant \max\big\{\mu d(x, x'), \mu^n d(y, y')\big\}. \tag{5}$$

For each $z = (x, y) \in Z$, there is only one orbit $(z_0, \dots, z_n)$ of length n of h such that x is the first component of z_0 and y the second component of z_n; in particular, the second component of z_{n-1} writes $y_{n-1} = A_{n-1}(z)$, defining a map $A_{n-1} : Z \to Y$ such that

$$d\big(A_{n-1}(z), A_{n-1}(z')\big) \leqslant \mu \max\big\{\lambda^{n-1} d(x, x'), d(y, y')\big\}. \tag{6}$$

Idea of the proof. If $n = 1$, this is true with $A_0 = G_1 = G$ and $F_1 = F$. Assuming it true for some $n \geqslant 1$, the sequence $(z_0, \dots, z_{n+1})$ is an orbit of length $n + 1$ of h if and only if, setting $z_j = (x_j, y_j)$ and $x_0 = x$, one has

$$x_n = F_n(x, y_n), \tag{7}$$

$$y_0 = G_n(x, y_n) \tag{8}$$

and, moreover, setting $y := y_{n+1}$,

$$\begin{aligned} x_{n+1} &= F(x_n, y), \\ y_n &= G(x_n, y). \end{aligned} \tag{9}$$

By (7), the last relation reads $y_n = G(F_n(x, y_n), y)$. For fixed (x, y), the Lipschitz constant of the right-hand side with respect to y_n is at most $\mu\lambda$ and therefore less than 1; hence, (9) is equivalent to

$$y_n = A_n(x, y) \tag{10}$$

[2] This title refers to the fact that the results of this section imply [1,2] the standard facts about *absolutely* normally hyperbolic invariant submanifolds. Examples and applications that we have no room to give here can be found in [2,1].

for a map $A_n : Z \to Y$, which is readily seen to satisfy (6) as required. The rest follows with

$$G_{n+1}(x,y) := G_n\big(x, A_n(x,y)\big) \quad \text{and} \quad F_{n+1}(x,y) := F\big(F_n\big(x, A_n(x,y)\big), y\big). \qquad \square \tag{11}$$

Corollary 1. *Under the hypotheses of Theorem* 1*, if Y is compact, then the set W_s of those $z \in Z$ such that there exists an orbit (z_n) of h with $z_0 = z$, namely*

$$W_s = \bigcap_{n \in \mathbf{N}} h^{-n}(Z), \tag{12}$$

is nonempty and has at least one point in $Z_x := \{x\} \times Y$ for every $x \in X$.

Proof. For each $x \in X$, Theorem 1 implies that $h^{-n}(Z) \cap Z_x$ is the nonempty compact subset consisting of all pairs $(x, G_n(x,y))$ with $y \in Y$. As $h^{-n}(Z)$ consists of those $z \in Z$ such that there exists an orbit $(z_0, \ldots, z_n)$ of h with $z_0 = z$, the subsets $h^{-n}(Z) \cap Z_x$ form a nonincreasing sequence of nonempty compact subsets, implying that their intersection $W_s \cap Z_x$ is nonempty. $\square$

From (11), we deduce at once

Corollary 2. *Under the hypotheses of Theorem* 1*, setting $A_{j,x}(y) := A_j(x,y)$, the maps G_n are obtained from the maps A_n by the formula*

$$G_n(x,y) = A_{0,x} \circ A_{1,x} \circ \cdots \circ A_{n-1,x}(y). \tag{13}$$

A sequence $(z_n)_{n\in\mathbf{N}}$ in Z is an orbit of h if and only if, setting $(z_n) = (x_n, y_n)$ and $x := x_0$, the relations (7)–(8) *or, equivalently,* (7)–(10) *hold for all n.*

Theorem 2. *Assume Y complete and $\mu < 1$. Then, for every $x \in X$ and $1 \leqslant \kappa < \mu^{-1}$, there exists a unique orbit $(x_n, y_n)_{n\in\mathbf{N}}$ of h such that $x_0 = x$ and that $(y_n)_{n\in\mathbf{N}}$ lies in the space $\mathcal{Y}_\kappa$ of those sequences $(y_n)_{n\in\mathbf{N}}$ in Y which satisfy $\sup_n \kappa^{-n} d(y_n, y) < \infty$ for some (and therefore all) $y \in Y$. For bounded Y, as $\mathcal{Y}_\kappa = Y^{\mathbf{N}}$, this is the unique orbit $(x_n, y_n)_{n\in\mathbf{N}}$ of h such that $x_0 = x$. Denoting y_0 by $\varphi(x)$, the map $\varphi : X \to Y$ has the following properties*:

(i) *$\varphi(x) = \lim_{n\to\infty} G_n(x,y)$ for every $y \in Y$ and all $x \in X$, hence* $\operatorname{Lip}\varphi \leqslant \mu$ *by* (5);
(ii) *the graph W_s of φ is invariant by h in the sense that $h^{-1}(W_s) = W_s$, and $W_s \ni z \mapsto h(z) \cap W_s$ is a Lipschitzian* map *$h_s : W_s \to W_s$ with* $\operatorname{Lip} h_s \leqslant \lambda$ *. When Y is bounded, W_s is the set* (12) *of those $z \in Z$ such that there exists an orbit (z_n) of h with $z_0 = z$.*

Proof. By Corollary 2, a sequence $(z_n)_{n\in\mathbf{N}} = (x_n, y_n)_{n\in\mathbf{N}}$ in Z with $x_0 = x$ is an orbit of h if and only if (7) holds for all n and the sequence $\mathbf{y} := (y_n)_{n\in\mathbf{N}}$ is a fixed point of the map $\mathcal{B}_x : \mathbf{y} \mapsto (A_n(x, y_{n+1}))_{n\in\mathbf{N}}$. Now, $\mathcal{B}_x$ is a strict contraction of $\mathcal{Y}_\kappa$ for the complete distance $d_\kappa(\mathbf{y}, \mathbf{y}') := \sup_n \kappa^{-n} d(y_n, y'_n)$, with $\operatorname{Lip}\mathcal{B}_x \leqslant \mu\kappa < 1$. It follows that $\mathcal{B}_x$ has a unique fixed point in $\mathcal{Y}_\kappa$, which is the first assertion of the theorem since (7) provides a *definition* of (x_n) from x and (y_n).

Proof of (i). For all $x \in X$ and $\mathbf{y} = (y_n) \in \mathcal{Y}_\kappa$, the y_0 component $\varphi(x)$ of the unique fixed point of $\mathcal{B}_x : \mathcal{Y}_\kappa \to \mathcal{Y}_\kappa$ is the y_0 component of $\lim_{n\to\infty} \mathcal{B}_x^{n+1}(\mathbf{y})$, namely, by (13), $\varphi(x) = \lim_{n\to\infty} G_n(x, y_n)$. Taking constant sequences, we get (i).

Proof of (ii). The identity $h^{-1}(W_s) = W_s$ is proved in [2]. Clearly, h_s is the map which associates to each $z = (x, \varphi(x)) \in W_s$ the z_1 term of the unique orbit (z_n) of h with $(y_n) \in \mathcal{Y}_\kappa$ such that $z_0 = z$. As the relation $z_1 \in W_s$ reads $y_1 = \varphi(x_1)$, the map h_s is of the form $h_s(x, \varphi(x)) = (\bar{h}_s(x), \varphi(\bar{h}_s(x)))$ and the inequality $\operatorname{Lip}\varphi < 1$ implies that $\operatorname{Lip} h_s = \operatorname{Lip} \bar{h}_s$. Now, the relation $(\bar{h}_s(x), \varphi(\bar{h}_s(x))) \in h(x, \varphi(x))$ yields $\bar{h}_s(x) = F(x, \varphi(\bar{h}_s(x)))$, hence, by (i) and since $\lambda = \operatorname{Lip} F$,

$$\begin{aligned} d\big(\bar{h}_s(x), \bar{h}_s(x')\big) &\leqslant \lambda \max\big\{d(x,x'), d\big(\varphi\big(\bar{h}_s(x)\big), \varphi\big(\bar{h}_s(x')\big)\big)\big\} \leqslant \max\big\{\lambda d(x,x'), \lambda\mu d\big(\bar{h}_s(x), \bar{h}_s(x')\big)\big\} \\ &\leqslant \lambda d(x,x') \end{aligned}$$

since $0 < (1 - \lambda\mu) d(\bar{h}_s(x), \bar{h}_s(x')) \leqslant 0$ is impossible, proving that $\operatorname{Lip} h_s \leqslant \lambda$. $\square$

Note. Generating maps, introduced by McGehee and Sander [7] to prove the stable manifold theorem, are used in [2], where the proof *à la Irwin* of (most of) Theorem 2 [3] is a little more involved analytically but avoids the combinatorics of Theorem 1. The advantage of the approach via Theorem 1 is that it works under the general (relative) normal hyperbolicity hypothesis of [4,6], as we shall now see.

3. The "relative" case in the C^r category

Hypothesis. We assume that X, Y are C^r Finsler manifolds with corners,[4] $r \geqslant 1$, that F, G are C^r and that there exist nonnegative constants λ, μ with

$$\lambda\mu < 1 \tag{14}$$

and functions $\alpha, \beta : Z \to [0, \infty)$ such that, for all $z = (x, y) \in Z$ and $\delta z = (\delta x, \delta y) \in T_z Z$,

$$\left|DF(z)\delta z\right| \leqslant \max\left\{\alpha(z)|\delta x|, \lambda|\delta y|\right\}, \tag{15}$$

$$\left|DG(z)\delta z\right| \leqslant \max\left\{\mu|\delta x|, \beta(z)|\delta y|\right\}, \tag{16}$$

$$\alpha(z)\beta(z) \leqslant 1. \tag{17}$$

The following analogue of Theorem 1 is proved exactly along the same lines:[5]

Theorem 3. *If Y is complete, then, for every positive integer n, the correspondence h^n has a C^r generating map $H_n = (F_n, G_n)$. Moreover, for all $z = (x, y) \in Z$,*

(i) *there exists a* unique *orbit $(z_0, \dots, z_n)$ of length n of h such that, setting $z_j = (x_j, y_j)$, one has $x_0 = x$ and $y_n = y$;*
(ii) *in particular, $y_{n-1} = A_{n-1}(z)$, defining a C^r map $A_{n-1} : Z \to Y$;*
(iii) *for all $\delta z = (\delta x, \delta y) \in T_z Z$, setting $v_j := (x_j, y_{j+1})$ for $0 \leqslant j \leqslant n-1$, one has*[6]

$$\left|DF_n(z)\delta z\right| \leqslant \max\left\{\alpha(v_{n-1}) \cdots \alpha(v_0)|\delta x|, \lambda|\delta y|\right\}, \tag{18}$$

$$\left|DG_n(z)\delta z\right| \leqslant \max\left\{\mu|\delta x|, \beta(v_0) \cdots \beta(v_{n-1})|\delta y|\right\}, \tag{19}$$

$$\left|DA_{n-1}(z)\delta z\right| \leqslant \max\left\{\mu\alpha(v_{n-2}) \cdots \alpha(v_0)|\delta x|, \beta(v_{n-1})|\delta y|\right\}. \tag{20}$$

Corollary 1 and Corollary 2 clearly hold in this new situation. Here is the analogue of Theorem 2:

Theorem 4. *Assume Y complete and $\beta_1 := \sup \beta(Z) < 1$. Then, for every $x \in X$ and $1 \leqslant \kappa < \beta_1^{-1}$, there exists a unique orbit $(x_n, y_n)_{n\in\mathbf{N}}$ of h such that $x_0 = x$ and that $(y_n)_{n\in\mathbf{N}}$ lies in the space $\mathcal{Y}_\kappa$ of those sequences $(y_n)_{n\in\mathbf{N}}$ in Y which satisfy $\sup_n \kappa^{-n} d(y_n, y) < \infty$ for some (and therefore all) $y \in Y$. For bounded Y, as $\mathcal{Y}_\kappa = Y^{\mathbf{N}}$, this is the unique orbit $(x_n, y_n)_{n\in\mathbf{N}}$ of h such that $x_0 = x$. Denoting y_0 by $\varphi(x)$, the map $\varphi : X \to Y$ has the following properties*:

(i) *$\varphi(x) = \lim_{n\to\infty} G_n(x, y)$ for every $y \in Y$ and all $x \in X$, hence* $\operatorname{Lip}\varphi \leqslant \mu$ *by* (19);
(ii) *the graph W_s of φ is invariant by h in the sense that $h^{-1}(W_s) = W_s$, and $W_s \ni z \mapsto h(z) \cap W_s$ is a locally Lipschitzian* map *$h_s : W_s \to W_s$, globally Lipschitzian for* $\sup\alpha(Z) < \infty$. *When Y is bounded, W_s is the set* (12) *of those $z \in Z$ such that there exists an orbit (z_n) of h with $z_0 = z$.*

The proof is analogous to that of Theorem 2. Smoothness of φ can be established for example as in [1] for $\sup_{z\in Z} \alpha(z)\beta(z) < 1$. As before, the analogues of almost all the results of [2] can be obtained in this more general setting. All this will be explained in a forthcoming article and in the book [3].

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3 Given as a sample: almost all the results of [2] can be revisited in the same spirit.

4 The Lipschitzian part of the theory obviously holds in the setting of Gromov's length structures [5].

5 The existence of the implicit function A_n follows from hypothesis (14), which clearly is satisfied in normal hyperbolicity results (the link is explained in [1,2]) since they deal with C^1-small perturbations of situations in which $\lambda = 0$.

6 Letting $\alpha(v_{n-2}) \cdots \alpha(v_0) = 1$ if $n = 1$.

Acknowledgements

This work owes much to conversations with Alain Chenciner. I am grateful to Albert Fathi, François Laudenbach, Santiago López de Medrano, Jean-Pierre Marco, Robert Moussu, Laurent Stolovitch and Eduard Zehnder for their encouragements, suggestions and remarks.

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