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Harmonic Analysis

Extended solution of Boas' conjecture on Fourier transforms

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Abstract

Weighted $L_p \to L_q$ Fourier inequalities are studied. We prove Boas' conjecture on integrability with power weights of the Fourier transform. One-dimensional as well as multidimensional versions (for radial functions) are obtained for general monotone functions. ***To cite this article: E. Liflyand, S. Tikhonov, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Solutions prolongées de la conjecture de Boas sur les transformées de Fourier. On étudie des inégalités $L^p \to L^q$ à poids pour des transformées de Fourier, en particulier on formule une conjecture de Boas traduisant une intégrabilité pour des fonctions dans le cas où le poids est une puissance lorsque l'une des fonctions est monotone et $p = q$. Nous donnons des versions unidimensionnelles et multidimensionnelles (dans le cas de fonctions radiales) pour $p \geqslant q$ ou $p \leqslant q$ et pour une classe définie de fonctions généralement monotones. ***Pour citer cet article : E. Liflyand, S. Tikhonov, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

L'intégrabilité des transformées de Fourier dont le poids est une fonction puissance impose des conditions sur les exposants. Le théorème de Pitt (voir [2]) implique l'inégalité :

$$\left[\int_{-\infty}^{\infty} \left|\hat{h}(x)\right|^p |x|^{-p\gamma}\,\mathrm{d}x\right]^{1/p} \leqslant C\left[\int_{-\infty}^{\infty} \left|h(t)\right|^q |t|^{q(1+\gamma-1/p-1/q)}\,\mathrm{d}t\right]^{1/q} \tag{1}$$

si $1 < q \leqslant p < \infty$ et $\max\{0, 1/p + 1/q - 1\} < \gamma < 1/p$; le résultat est faux si γ ne vérifie pas ces dernières inégalités. Dans toute la suite C est une constante absolue qui peut prendre différentes valeurs. Pour élargir le champ d'application on doit imposer des restrictions supplémentaires à h.

Pour les fonctions monotones R.P. Boas [3] a formulé la conjecture suivante :

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Soit $H(x)=\int_0^\infty h(t)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t$, $x\geqslant 0$, alors, on a :

Conjecture. *Si* $1<p<\infty$ *et h non negative et monotone sur* $(0,\infty)$*, alors* $x^{-\gamma}H(x)\in L^p(0,\infty)$ *si et seulement si* $t^{1+\gamma-2/p}h(t)\in L^p(0,\infty)$ *sous la condition* $-1/p'=-1+1/p<\gamma<1/p$.

Définition. [6] (cf. [5]) Soit h une fonction localement à variation bornée sur $(0,\infty)$, nulle à l'infini ; on dira qu'elle est « généralement monotone », *GM*, si pour tout $x\in(0,\infty)$ et pour une constante $c>1$, on a :

$$\int_x^{2x}\left|dh(t)\right|\leqslant C\int_{x/c}^{cx}u^{-1}\left|h(u)\right|\mathrm{d}u.$$

Le théorème suivant est le résultat fondamental de cette Note :

Théorème 0.1. *Soient* $1\leqslant p,q<\infty$ *avec* $-1/p'<\gamma<1/p$. *Soit h une fonction non negative and GM. On a*

(A) *Si* $q\leqslant p$, *alors* $x^{1+\gamma-1/p-1/q}h(x)\in L^q(0,\infty)$ *implique* $x^{-\gamma}H(x)\in L^p(0,\infty)$;
(B) *Si* $p\leqslant q$, *alors* $x^{-\gamma}H(x)\in L^p(0,\infty)$ *implique* $x^{1+\gamma-1/p-1/q}h(x)\in L^q(0,\infty)$.

Dans le cas $p=q$, la conjecture de Boas est vraie pour toute fonction h monotone et *GM*.

1. Introduction

Integrability of the Fourier transform with power weights induces natural constraints on the exponents. Pitt's theorem (see, e.g., [2]) asserts that (1) holds if $1<q\leqslant p<\infty$ and $\max\{0,1/p+1/q-1\}<\gamma<1/p$; the result fails for γ outside this range. Here and below C will denote various absolute constants. To widen the range, additional restrictions on h should be posed. For instance, vanishing of certain moments is assumed in [7]. For monotone functions, R.P. Boas [3] conjectured the following. Let $H(x)=\int_0^\infty h(t)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t$, $x\geqslant 0$.

Conjecture 1.1. *If* $1<p<\infty$ *and h is non-negative and monotone on* $(0,\infty)$*, then* $x^{-\gamma}H(x)\in L^p(0,\infty)$ *if and only if* $t^{1+\gamma-2/p}h(t)\in L^p(0,\infty)$ *provided* $-1/p'=-1+1/p<\gamma<1/p$.

2. The (p,q) Fourier inequalities

We will prove an enriched version of Boas' conjecture. First, we consider weighted integrability of a function and its Fourier transform in different Lebesgue spaces, that is, in L^p and L^q. Next, the result is obtained for a wider class of h being *general monotone* functions. For $p=q$ it is exactly the assertion of the conjecture but for a wider class of general monotone functions.

This notion is introduced in [6] as an analog of *general monotonicity* brought in for sequences in [10]. We call a function *admissible* if it is locally of bounded variation on $(0,\infty)$ and vanishes at infinity. The following definition is given in [6] (cf. [5]):

Definition. We call an admissible function h general monotone, written *GM*, if for all $x\in(0,\infty)$,

$$\int_x^{2x}\left|\mathrm{d}h(t)\right|\leqslant C\int_{x/c}^{cx}u^{-1}\left|h(u)\right|\mathrm{d}u, \tag{2}$$

for some $c>1$.

Note that any monotone (or quasi-monotone) function belongs to *GM*. We need the following two properties of the *GM* functions (see [6,11]):

$$\left|h(x)\right| \leqslant C \int_{x/c}^{cx} u^{-1}\left|h(u)\right| \mathrm{d}u, \quad x>0 \tag{3}$$

and

$$\int_{x}^{y}\left|\mathrm{d}h(t)\right| \leqslant C \int_{x/c}^{cy} u^{-1}\left|h(u)\right| \mathrm{d}u, \quad 0<x<y. \tag{4}$$

We will assume h to be integrable near the origin. Since h is admissible, it is locally integrable on $\mathbb{R}_+$ and H is well defined in the distributional sense. Our main result reads as follows:

Theorem 2.1. *Let $1 \leqslant p, q<\infty$ and $-1/p'<\gamma<1/p$. Assume h to be non-negative and GM. Then,*

(A) *If $q \leqslant p$, then $x^{1+\gamma-1/p-1/q} h(x) \in L^q(0, \infty)$ implies $x^{-\gamma} H(x) \in L^p(0, \infty)$;*
(B) *If $p \leqslant q$, then $x^{-\gamma} H(x) \in L^p(0, \infty)$ implies $x^{1+\gamma-1/p-1/q} h(x) \in L^q(0, \infty)$.*

Proof. We begin with the upper estimate of $\|x^{-\gamma} H(x)\|_{L^p}$. Let $\Phi(x):=\int_x^{2x}|\mathrm{d}h(t)|\,\mathrm{d}t$.

Lemma 2.2. *There holds*:

$$\left|H(x)\right| \leqslant C\left(\int_0^{\pi/x} \Phi(t)\,\mathrm{d}t+x^{-1} \int_{\pi/x}^{\infty} t^{-1} \Phi(t)\,\mathrm{d}t\right). \tag{5}$$

Proof. Splitting the integral and integrating by parts, we obtain:

$$\left|H(x)\right|=\left|\int_0^{\pi/x} h(t) \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t+(\mathrm{i}x)^{-1} h(\pi/x)-(\mathrm{i}x)^{-1} \int_{\pi/x}^{\infty} \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}h(t)\right| \leqslant \int_0^{\pi/x}\left|h(t)\right|+2x^{-1} \int_{\pi/x}^{\infty}\left|\mathrm{d}h(t)\right|.$$

Since $\int_0^{\pi/x}|h(t)|\,\mathrm{d}t \leqslant \int_0^{\pi/x}\int_t^{\pi/x}|\mathrm{d}h(s)|\,\mathrm{d}t+\int_0^{\pi/x}\int_{\pi/x}^{\infty}|\mathrm{d}h(s)|\,\mathrm{d}t$, we have:

$$\left|H(x)\right| \leqslant C \int_0^{\pi/x} s\left|\mathrm{d}h(s)\right|+\int_{\pi/x}^{\infty}\left|\mathrm{d}h(s)\right|. \tag{6}$$

Applying now, with ψ being any integrable function,

$$\int_0^B t^{-1} \int_t^{2t}\left|\psi(u)\right| \mathrm{d}u\,\mathrm{d}t \geqslant \ln 2 \int_0^B\left|\psi(u)\right| \mathrm{d}u \quad \text{and} \quad \int_A^{\infty} t^{-1} \int_t^{2t}\left|\psi(u)\right| \mathrm{d}u\,\mathrm{d}t \geqslant \ln 2 \int_{2A}^{\infty}\left|\psi(u)\right| \mathrm{d}u,$$

to the right-hand side of (6), we complete the proof of the lemma. □

We will use the following extension [4] of Hardy's inequality: if $1 \leqslant \alpha \leqslant \beta<\infty$, then

$$\left[\int_0^{\infty} u(x)\left(\int_0^x \psi(t)\,\mathrm{d}t\right)^\beta \mathrm{d}x\right]^{1/\beta} \leqslant C\left[\int_0^{\infty} v(x) \psi(x)^\alpha\,\mathrm{d}x\right]^{1/\alpha} \tag{7}$$

holds for every $\psi, u, v \geqslant 0$ if and only if $\sup_{t>0}(\int_t^{\infty} u(x)\,\mathrm{d}x)^{1/\beta}(\int_0^t v(x)^{1-\alpha'}\,\mathrm{d}x)^{1/\alpha'}<\infty$.

Let $p \geqslant q$. Estimating the first integral on the right in (5), we wish to have:

$$\left[\int_0^\infty x^{p\gamma-2}\left|\int_0^x \Phi(t)\,\mathrm{d}t\right|^p \mathrm{d}x\right]^{1/p} \leqslant C\left[\int_0^\infty x^{q\gamma+q-1-q/p}\Phi(x)^q\,\mathrm{d}x\right]^{1/q}.$$

Straightforward calculations show that this holds when $\gamma < 1/p$.

Estimating now the second integral on the right in (5), we obtain, by substituting $t \to \pi/t$ and applying (7) with appropriate power weights,

$$\left[\int_0^\infty x^{-p\gamma-p}\left|\int_{\pi/x}^\infty t^{-1}\Phi(t)\,\mathrm{d}t\right|^p \mathrm{d}x\right]^{1/p} \leqslant \left[\int_0^\infty x^{-p\gamma-p}\left|\int_0^x t^{-1}\Phi(\pi/t)\,\mathrm{d}t\right|^p \mathrm{d}x\right]^{1/p}$$
$$\leqslant C\left[\int_0^\infty x^{-q\gamma-1+q/p}\big(x^{-1}\Phi(\pi/x)\big)^q\,\mathrm{d}x\right]^{1/q}.$$

This holds only when $\gamma > -1/p'$.

Since h is *GM*, using (4) along with Hardy's inequality (7) with $\alpha = \beta = q$ yields:

$$\left[\int_0^\infty \left|x^{-\gamma}H(x)\right|^p \mathrm{d}x\right]^{1/p} \leqslant C\left[\int_0^\infty u^{q\gamma+q-1-q/p}\left(\int_{u/c}^{cu} s^{-1}h(s)\,\mathrm{d}s\right)^q \mathrm{d}u\right]^{1/q}$$
$$\leqslant C\left[\int_0^\infty u^{q\gamma-1-q/p}\left(\int_0^u h(s)\,\mathrm{d}s\right)^q \mathrm{d}u\right]^{1/q} \leqslant C\left[\int_0^\infty u^{q\gamma+q-1-q/p}h(u)^q\,\mathrm{d}u\right]^{1/q},$$

with $\gamma < 1/p$. This proves the first part of the theorem.

To prove the part (B), we need appropriate lower estimates for $q \geqslant p$. Letting $H_s(x) = \int_0^\infty h(t)\sin xt\,\mathrm{d}t$ to be the sine Fourier transform of h, we have $\int_0^u H_s(x)\,\mathrm{d}x = 2\int_0^\infty h(t)t^{-1}\sin^2(ut/2)\,\mathrm{d}t \geqslant 0$. We then obtain, with c from (3),

$$\int_0^{\pi/(2cx)} \left|H(x)\right|\mathrm{d}x \geqslant \int_0^{\pi/(2cx)} \left|H_s(x)\right|\mathrm{d}x \geqslant \left|\int_0^{\pi/(2cx)} H_s(x)\,\mathrm{d}x\right| \geqslant C\int_{x/c}^{cx} t^{-1}h(t)\,\mathrm{d}t;$$

then

$$\left[\int_0^\infty\left(\int_{t/c}^{ct} s^{-1}h(s)\,\mathrm{d}s\right)^q t^{q\gamma+q-1-q/p}\,\mathrm{d}t\right]^{1/q} \leqslant C\left[\int_0^\infty t^{q\gamma+q-1-q/p}\left(\int_0^{\pi/(2ct)} \left|H(u)\right|\mathrm{d}u\right)^q \mathrm{d}t\right]^{1/q}.$$

Substituting $\pi/(2ct) = x$ and applying (7), we obtain:

$$\left[\int_0^\infty x^{-q\gamma-q-1+q/p}\left(\int_0^x \left|H(u)\right|\mathrm{d}u\right)^q \mathrm{d}x\right]^{1/q} \leqslant C\left(\int_0^\infty x^{-p\gamma}\left|H(x)\right|^p \mathrm{d}x\right)^{1/p},$$

which is true if and only if $\gamma > -1/p'$. Applying (3), we have

$$\left[\int_0^\infty \left|h(t)t^{1+\gamma-1/p-1/q}\right|^q \mathrm{d}t\right]^{1/q} \leqslant C\left[\int_0^\infty \left|H(x)x^{-\gamma}\right|^p \mathrm{d}x\right]^{1/p}, \tag{8}$$

which completes the proof. □

We note that part (B) of Theorem 1 holds if only $-1/p' < \gamma$.

3. One-dimensional Boas-type result

In the case $p = q$ we have a direct generalization of Boas' conjecture.

Corollary 1. *Let $1 \leqslant p < \infty$. If h is non-negative and general monotone on $(0, \infty)$, then*

$$x^{-\gamma} H(x) \in L^p(0,\infty) \quad \text{if and only if} \quad t^{1+\gamma-2/p} h(t) \in L^p(0,\infty),$$

provided $-1/p' < \gamma < 1/p$.

Note that for monotone h this result was proved by Sagher [8]. We also remark that our estimates of $\|x^{-\gamma} H(x)\|_p$ are similar to known estimates for sequences by Askey and Wainger [1].

We have considered the Fourier transform H to directly fit Boas' conjecture, since the latter does not distinct between cosine and sine transforms. It turns out that the difference between them is hidden in such a way. Define, in addition, the cosine Fourier transform of h by $H_c(x) = \int_0^\infty h(t)\cos xt\,\mathrm{d}t$. It follows from Corollary 1 that for non-negative $h \in GM$,

$$x^{-\gamma} H_c(x) \in L^p(0,\infty) \quad \text{if and only if} \quad t^{1+\gamma-2/p} h(t) \in L^p(0,\infty),$$

for $-1/p' < \gamma < 1/p$. However, the counterpart for H_s holds for the wider range $-1/p' < \gamma < 1 + 1/p$.

Indeed, estimate (8), actually fulfilled for H_s, provides the part "only if" for $-1/p' < \gamma$. To improve $\gamma < 1/p$, we consider an appropriate version of Lemma 2.2 for the sine transform alone:

$$\bigl|H_s(x)\bigr| \leqslant C\left(x\int_0^{\pi/x} t\Phi(t)\,\mathrm{d}t + x^{-1}\int_{\pi/x}^{\infty} t^{-1}\Phi(t)\,\mathrm{d}t\right),$$

which could be shown similarly to the proof of the lemma by using $|\sin(xt)| \leqslant |xt|$. In addition, here we may assume a weaker condition of integrability of $th(t)$ near the origin. Further, by Hardy's inequality,

$$\left[\int_0^\infty \left|x^{1-\gamma}\int_0^{\pi/x} t\Phi(t)\,\mathrm{d}t\right|^p \mathrm{d}x\right]^{1/p} \leqslant C\left[\int_0^\infty u^{\gamma p+p-2}\Phi^p(u)\,\mathrm{d}x\right]^{1/p},$$

just with $\gamma < 1 + 1/p$. The rest is the same.

We finally mention here that $|x|^{-\gamma p}$ belongs to the Muckenhoupt A_p class for $-1/p' < \gamma < 1/p$. Therefore, in the case of sine transform we can deal not only with Muckenhoupt weights.

4. Boas-type result for radial functions

Let us now make use of obtained results – for simplicity, Corollary 1 – in problems of integrability of the multidimensional Fourier transform of a radial function $f(x) = f_0(|x|)$. Let $\hat{f}$ denote its usual Fourier transform on $\mathbb{R}^n$. It is known (see Lemma 25.1′ in [9]) that if,

$$\int_0^\infty t^{n-1}(1+t)^{(1-n)/2}\bigl|f_0(t)\bigr|\,\mathrm{d}t < \infty, \tag{9}$$

the following relation holds,

$$\hat{f}(x) = 2\pi^{(n-1)/2}\int_0^\infty I(t)\cos|x|t\,\mathrm{d}t, \tag{10}$$

where the fractional integral I is given by:

$$I(t) = \frac{2}{\Gamma(\frac{n-1}{2})}\int_t^\infty sf_0(s)\bigl(s^2 - t^2\bigr)^{(n-3)/2}\,\mathrm{d}s.$$

Theorem 4.1. *Let f_0 satisfy* (9)*, while I be non-negative and GM on* $(0,\infty)$*, then*

$$|x|^{-\gamma}\hat{f}(x)\in L^p(\mathbb{R}^n) \quad \text{if and only if} \quad t^{1+\gamma-(n+1)/p}I(t)\in L^p(0,\infty),$$

provided $-1+n/p<\gamma<n/p$.

To prove this, let us multiply the absolute value of the right-hand side of (10) by $|x|^{-\gamma}$ and estimate its L^p norm over $\mathbb{R}^n$. Using polar coordinates, we get, up to some constant,

$$\left(\int_0^\infty u^{-p(\gamma-(n-1)/p)}\left|\int_0^\infty I(t)\cos ut\,\mathrm{d}t\right|^p \mathrm{d}u\right)^{1/p}.$$

By Corollary 1 with $\gamma-(n-1)/p$ in place of γ, we immediately obtain the required result.

Remark 1. We observe that for $n=1$ understanding I formally as f_0 reduces Theorem 4.1 to Corollary 1. Nevertheless, this does not ensures optimality of the range of involved parameters. Indeed, in the case of $\mathbb{R}^3$ the Fourier transform (if exists) of a radial function is of especially simple form $\hat{f}(x)=4\pi|x|^{-1}\int_0^\infty tf_0(t)\sin|x|t\,\mathrm{d}t$.

Applying the result for the sine transform to the function $tf_0(t)$, we derive that $\hat{f}(x)|x|^{-\gamma}\in L^p(\mathbb{R}^3)$ if and only if $t^{3+\gamma-4/p}f_0(t)\in L^p(0,\infty)$ provided $-2+3/p<\gamma<3/p$. Note that we assume only the general monotonicity of $f_0(t)$, since this implies that $tf_0(t)$ is general monotone as well.

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