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Rigorous construction of the pure states for certain spin glass models

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Abstract

For a large class of Gaussian Hamiltonians, that includes the Sherrington–Kirkpatrick model, the p-spin interaction model for even p, and many others, we prove at each temperature (under a very mild and essentially necessary condition) that generically the "pure states" predicted by physics do exist. The configuration space decomposes in an essentially unique manner in a sequence of subsets on which the overlap takes essentially its maximum value. ***To cite this article: M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Construction rigoureuse des états purs pour certains modèles de verre de spin. Pour une large classe d'Hamiltoniens Gaussiens, comprenant le modèle de Sherrington–Kirkpatrick, le modèle à p-spin pour p pair, et bien d'autres, nous prouvons à toute temperature (sous des conditons très peu restrictives) l'existence des « états purs » prédits par les physiciens. Génériquement, à désordre donné, l'espace des configurations se décompose de façon essentiellement unique en une suite de sous-ensembles sur lesquels le recouvrement de deux configurations prend essentiellement sa valeur maximale. ***Pour citer cet article : M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Statement of the results

In this Note we consider the following class of Gaussian Hamiltonians. Consider, for $p \geqslant 1$ and all integers $i_1, \dots, i_p$ independent standard Gaussian r.v. $g_{i_1 \dots i_p}$. Given numbers β_p for $p \geqslant 1$, and given $\boldsymbol{\sigma} \in \mathbb{R}^N$, define

$$-H_N(\boldsymbol{\sigma}) = \sum_{p \geqslant 1} \frac{\beta_p}{N^{(p-1)/2}} \sum g_{i_1 \dots i_p} \sigma_{i_1} \cdots \sigma_{i_p}. \tag{1}$$

In this equation the second summation is over all values of $i_1, \dots, i_p \leqslant N$. The first summation is over the values of $p = 1$ or p even. We expect that the results we present are also correct if we include in the summation terms for $p \geqslant 3$, p odd, but our approach relies on previous results that remain unproven in this case. The Hamiltonian considered in (1) induces a Gibbs measure on the set of all configurations. This set is usually $\Sigma_N = \{-1, 1\}^N$ (the Ising model) or

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$S_N = \{\boldsymbol{\sigma} \in \mathbb{R}^N; \sum_{i \leqslant N} \sigma_i^2 = N\}$ (the spherical model). The Ising case includes in particular the famous Sherrington–Kirkpatrick (SK) model. The Gibbs measure is simply the probability measure on the space of configurations with density proportional to $\exp(-H_N)$ with respect to the uniform measure. (One can also add an external field without changing the conclusion of our discussion.) The structure of this Gibbs measure is a question of central importance.

The structure of the Gibbs measure for the SK model is described in [1] from the physicists' point of view. This description of the system starts with the notion of "pure states", and the existence of these seem to be taken for granted. Moreover, according to a discussion with M. Mézard, the point of view in [1] is "to take $N = \infty$ to help the intuition" (and this despite the fact that it does not seem possible to give a mathematical meaning to this situation). That leaves open the question of giving a mathematically meaningful definition of these "pure states" and a proof of their existence. The contribution of this note is to explain why this is a consequence of some previous results, together with a new and (hopefully) non-trivial structure result about certain probability measures on Hilbert space.

A central idea is the notion of the overlap of two configurations $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$, which is defined by $R_{1,2} = N^{-1} \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2$. Observe that, within the different normalization, this is simply the usual dot product in $\mathbb{R}^N$ of $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$. In physics terms, the "pure states" have the property that the overlap is "generically constant" over them. A small part of the physical picture is then as follows. At a given disorder, the configuration space decomposes in a sequence of "pure states". Given any two "generic" configurations in the same pure state, their overlap is a certain number q^*, that is independent both of this pure state and, remarkably, of the disorder. Given any two configurations in two different pure states A, B, their overlap is generically $< q^*$. This is the part of the picture that we will prove here, although of course since we do not deal with the ideal case $N = \infty$ the statement has to be technically more complicated. The full picture from the physicists involve a "ultrametric organization" of these pure states, the rigorous proof of which being the main remaining problem of the area.

A simple observation is as follows. Given the previous picture (that we still describe "at $N = \infty$"), if we consider the law of the overlap under Gibbs' measure, it has point mass at q^*. The mass at this point is exactly the sum of the squares of the weights of the pure states under Gibbs' measure. Therefore (the limit of) the average ν_N of this law under the disorder, that we will call the Parisi measure, must have a point mass at q^*, and q^* will be the largest point of its support.

The existence of the Parisi measure (defined as the limit of ν_N as $N \to \infty$) is proved in many situations in [3] for the Ising case and in [4] for the Spherical case. The Parisi measure is described as the minimizer of a rather complicated functional, and its actual computation is very difficult, specially in the Ising case. Still, an explicit example is constructed in [4] where this Parisi measure *does not* have a point mass at the largest point of its support. For this system, the "pure states picture" cannot be literally true. Its seems, however, (based on the numerical simulations of the physicists), that this situation is exceptional, and that "typically" the Parisi measure does have a positive mass at the largest point of its support (it is an open problem to prove a rigorous form of this statement). Therefore its make sense to prove that the "pure states picture" holds under this hypothesis, which is in any case necessary. This is what the present work does.

2. A theorem on measures on Hilbert space

We consider a separable Hilbert space, and for two vectors x and y we denote by $x \cdot y$ their dot product.

Theorem 2.1. *Given the number $\alpha > 0$, there exists a number ε with the following property. Consider a probability measure μ on the unit ball of H. Assume that for a certain number q^* we have*

$$\mu^{\otimes 2}\big(\big\{(x, y); x \cdot y \geqslant q^* + \varepsilon\big\}\big) \leqslant \varepsilon. \tag{2}$$

Then we can find disjoint sets $A_1, \ldots, A_k$ with the following properties:

$$\forall \ell \leqslant k, \quad \int_{A_\ell^2} |x \cdot y - q^*| \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \leqslant \alpha \mu(A_\ell)^2. \tag{3}$$

$$\mu^{\otimes 2}\bigg(\big\{(x, y); |x \cdot y - q^*| \leqslant \varepsilon\big\} \Big\backslash \bigcup_{\ell \leqslant k} A_\ell^2\bigg) \leqslant \alpha. \tag{4}$$

This theorem can be explained as follows. Condition (2) means that within the small error ε, the support of the law ν of $\mu^{\otimes 2}$ under the map $(x, y) \mapsto x \cdot y$ is located to the left of the point q^*. The case of interest is when the set

$$U = \{(x, y); x \cdot y \geqslant q^* - \varepsilon\} \tag{5}$$

is not too small for $\mu^{\otimes 2}$, for otherwise it suffices to take $k = 0$. The condition that this set is not too small means that within error ε, ν has a significant mass at the point q^*.

The sets A_ℓ are the "pure states" we are looking for. Condition (3) means that within the error level α, the "overlap" (= dot product) of any two points in one of these pure states is equal to q^*, and condition (4) that within the same error level, the only way the overlap of any two points can be equal to q^* (within error ε) is that they belong to the same pure state. Moreover, the error level α with which we perform this construction can be made as small as possible by taking ε small enough. What the theorem does not say is that the sets A_ℓ exhaust the entire measure μ; but of course this is not true under these hypotheses.

For the application to spin glasses we need only Hilbert spaces of finite dimension. The important point of course is that the dependence of ε on α does not involve this dimension.

The theorem is true in the case $\alpha = \varepsilon = 0$, but in that case it is essentially trivial. The sets A_ℓ are reduced to single points. The argument goes as follows. To avoid an even more trivial situation we assume that $q^* > 0$. Consider the number $c \geqslant 0$ such that the ball B_c centered at 0 of radius c is the smallest such ball that supports μ. If $d < c$, then the ball B_d is such that $\mu(B_c \setminus B_d) > 0$. The set $B_c \setminus B_d$ is a countable union of sets of the type $S(y) = \{x; x \cdot y \geqslant d\}$ where y is a vector of norm one. Thus one of these sets has a positive measure for μ. Since the dot product of any two vectors in $S(y) \cap B_d$ is nearly c^2, this shows that c^2 is in the support of μ, and therefore that $c^2 = q^*$. Then the only pairs (x, y) in B_c such that $x \cdot y = q^*$ are those for which $x = y$, and all the mass of ν at q^* comes from such pairs. It therefore comes from atoms of μ located on the boundary of B_c, and the sum of the squares of the masses of these atoms is $\nu(q^*)$.

Apparently simple measure theoretic arguments do not suffice to prove Theorem 2.1. Is seems that quantitative arguments (in the form of Sudakov minoration in our proof) are required. The complete details of this proof will appear in the forthcoming book [5]. A weaker (and much easier) form of Theorem 2.1 was proved in Chapter 6 of [2] in the case where ν is almost supported by the points 0 and q^*, where q^* is very close to 1.

3. Application to spin glasses

Assuming that the Parisi measure of a given Hamiltonian (1) has a point mass at the largest point q^* of its support, consider an error level α, and ε as provided by Theorem 2.1. Then for N large and for most of the realizations of the randomness, the Gibbs' measure (after the obvious normalization by the factor $\sqrt{N}$) will satisfy condition (2), so Theorem 2.1 immediately constructs the "pure states". The one part missing for our statement in the introduction is that (generically) the pure states essentially exhaust the entire space of configurations. The reason for this is different. It is the fact that the Ghirlanda–Guerra identities imply that the sequence of weights of the "pure states" constructed by Theorem 2.1 has asymptotically a Poisson–Dirichlet distribution, so that the sum of these weights much be nearly one. It is a bit complicated to give a mathematically precise statement, because the Ghirlanda–Guerra identities are not true in general, but only "in average", or if one adds a suitable infinitesimal perturbation term to the Hamiltonian. This is what the term "generically" tries to convey. We refer the reader to Chapter 6 of [2] for a more complete discussion of this idea, to which we have nothing to add at this point.

References

[1] M. Mézard, G. Parisi, M. Virasoro, Spin Glass Theory and Beyond, World Scientific, Singapore, 1987.
[2] M. Talagrand, Spin Glasses, A Challenge to Mathematicians, Springer-Verlag, 2003.
[3] M. Talagrand, Parisi measures, J. Funct. Anal. 231 (2) (2006) 269–286.
[4] M. Talagrand, Free energy of the spherical mean field model, Probab. Theory Related Fields 134 (3) (2006) 339–382.
[5] M. Talagrand, Mean Field Models for Spin Glasses, book in preparation.